An Adventure in the Realm of Topology

Mohammad Sal Moslehian

ELEMENT

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Dedicated to Professor Bahman Honary,
with utmost respect and gratitude.

Foreword

Euclidean geometry can be thought of as the study of figures with a rigid shape. Topology, also known as rubber sheet geometry, studies the shape of objects up to continuous deformations that can be continuously undone.

Writing a nontechnical book on topology is a difficult task. If, in addition, the author intends to address teenagers, the difficulty increases.

The author's experience with teaching this subject to teenagers is essential for writing an attractive and challenging text. This is the case of Professor Mohammad Sal Moslehian. Supported by his experience, not only in university teaching, but also in training teachers, he offers us a textbook, with quite advanced topics and many activities, that will certainly engage students, and create an interest in new concepts.

The book contains many activities, nicely illustrated, and these will develop the natural tendency in teenagers to be explorers. For instance, in one activity the reader is asked to use a marker to draw a circle, a triangle, and a square on an uninflated balloon and then inflate the balloon. Then the reader is asked to explain why the resulting shapes are topologically equivalent to the original ones.

The book also contains many challenging topics: four-dimensional spaces, knot theory, and fractals, to mention a few. The formal study of these topics requires many prerequisites. However, with his ample experience, Professor Moslehian manages to communicate them without technicalities, engaging the curious students in a pleasant ludic journey. The book will appeal to anyone interested in geometry and can be regarded as a great contribution to the popularization of mathematics.

Ton Marar
University of São Paulo, Brazil
February 2024

Preface

Throughout the years, I have taught general topology to undergraduates in universities, gained experience teaching high school students, and conducted research in related areas of functional analysis, specifically topological algebras. I believe that teenagers can grasp the fundamentals of topology while studying geometry. It is worth noting that others have also shared this belief and presented arguments in its favor.

Based on this conviction, I have been training high school teachers and my Ph.D. students on this subject for two decades. Now, I am summarizing my findings and experiences in this book to assist high school and university students in understanding complicated topological concepts.

Topology is typically introduced at higher education levels. However, due to the visual essence of topology, it is feasible to teach high school students some basic concepts and help them develop a deeper understanding of these concepts through challenging yet intuitive examples and activities.

While this book is designed for self-study and is a simplified approach to basic topology by avoiding the Theorem-Proof style, its most effective use is in the hands of experienced teachers.

It is crucial to make topology fun and age-appropriate in order to effectively engage students. To foster a deeper understanding of topology, it is important to encourage students to explore and experiment with shapes and objects. One way to do this is by providing them with play dough and asking them to mold different shapes such as spheres, pyramids, cups, and donuts. By manipulating these shapes, students can gain a better understanding of topological concepts.

I would like to sincerely thank Professor Rahim Zaare-Nahandi, Professor Fateme Helen Ghane, Professor Majid Mirzavaziri, and Dr. Zohreh Vasagh for their valuable feedback. I also appreciate Professor Ton Marar for graciously writing a foreword for the book.

I am grateful to Mr. Hossein Eslamimanesh for creating the figures in this book using various software tools, to Ms. Mobina Gohari for utilizing an AI-based tool to generate the photo on the cover, and to my daughter, Dr. Anahita Sal Moslehian for designing the cover of the book.

Finally, special thanks go to Professor Neven Elezović, from Element Publishing House, for making this book freely accessible to everyone. I would like to express my sincere appreciation to the editor of the book, Ms. Sandra Gračan, and the designer, Ms. Lejla Bužinkić for their exceptional efforts in creating such an elegant final product.

M. S. Moslehian
April 2024

Introduction

We can transform a cup into a donut without cutting or gluing. However, a cup and a tennis ball cannot undergo such a topological transformation[1] because it would cause the cup's handle to break. Topology, a branch of mathematics, aims to formalize and explore these similarities and differences, extending them to abstract spaces. It investigates the properties of geometric entities that remain unchanged (invariant) under topological transformations, such as torsion, stretching, and bending, while excluding actions like creating holes, gluing, tearing, and self-intersection. Topology seeks to define the concept of "closeness" without relying on the term "distance".

The term "topology" is derived from the Greek words "topo" (place) and "logy" (study). Gottfried Wilhelm Leibniz (1646–1716) was among the first mathematicians to deal with ideas related to topology. Leonhard Euler (1707–1783) can be credited with laying the foundation for topology through his work on the problem of the "Seven Bridges of Königsberg". The first book on topology, titled "Preliminary Studies on Topology", was written by Johann Benedict Listing (1808–1882), who coined the term "topology". The distinct field of topology in mathematics is often traced back to the publication of "Analysis Situs" by the French mathematician Henri Poincaré (1854–1912) in 1895. Other significant contributors include August Ferdinand Möbius (1790–1868), Bernhard Riemann (1826–1866), Camille Jordan (1838–1922), and Felix Hausdorff (1868–1942). Throughout the 20th and 21st century, topology has continued to develop with contributions from mathematicians worldwide.

Topology is concerned with the qualitative properties of space that remain unchanged in topological transformations, focusing on concepts such as connectivity and compactness. In contrast, geometry focuses on the quantitative aspects of space and deals with precise measurements such as distance, length, area, volume, and angle.

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[1]It is also known as continuous deformation which refers to a one-to-one correspondence that is continuous and its inverse is also continuous

Teaching basic concepts of topology requires simple yet engaging examples that provide a foundation for future mathematical concepts. The book aims to increase accessibility by providing simple examples and visuals that match students' abilities and spark their interest in exploring complex ideas.

Encouraging students to learn about topology can be challenging. However, with the right approach, we can foster interest and curiosity in the properties of shapes and spaces. As they progress in their education, they can explore more advanced topological concepts.

This book is self-contained, making it easy for teachers to instruct high school and university students. The book meticulously presents the basic concepts of topology and enhances students' understanding of geometry.

In Chapter 1, we provide teachers with guidelines for teaching basic topology to students. They should remember that the key to engaging students in topology is using age-appropriate and varied teaching methods, as well as making tangible connections to the real world in a supportive learning environment. This approach can make the subject engaging and fun. This chapter concludes with a variety of easy activities for beginners.

In Chapter 2, we focus on introducing learners to the basic concepts of set theory in a clear and engaging manner.

In Chapter 3, we examine central concepts of topology such as curves, neighborhoods, types of points, open and closed sets, boundedness, connectedness, and compactness, all explained in a user-friendly way.

In Chapter 4, we explore the ideas of topological equivalence, topological invariants, graphs, Euler characteristics, holes, and handles. In particular, we use topological invariants to demonstrate that two shapes or bodies are not topologically equivalent.

In Chapter 5, we introduce topics such as four-dimensional space, knot theory, and fractals. While these topics can be challenging for students, they become engaging when presented in a way that ignites their curiosity and desire to explore.

The book is enriched with captivating images and includes a variety of activities designed to stimulate students' thinking. The activities are categorized as simple, moderate, and challenging, indicated by green, yellow, and red boxes, respectively:

> **Activity 0.1**
>
> This is an simple activity.

> **Activity 0.2**
>
> This is a moderate activity.

> **Activity 0.3**
>
> This is a challenging activity.

Furthermore, the sections are specified as simple, moderate, or challenging by the following symbols.

Figure 1. Simple Topic

Figure 2. Moderate Topic

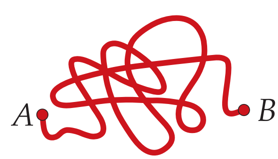


Figure 3. Challenging Topic

In addition, the book features readings that encourage readers to explore complex topics. Moreover, a comprehensive list of references is provided, which includes books and articles for further reading. The book itself, along with the interesting book [12] and the papers [6, 18], serve as valuable resources for developing mathematics lessons in modern pedagogy.

Contents

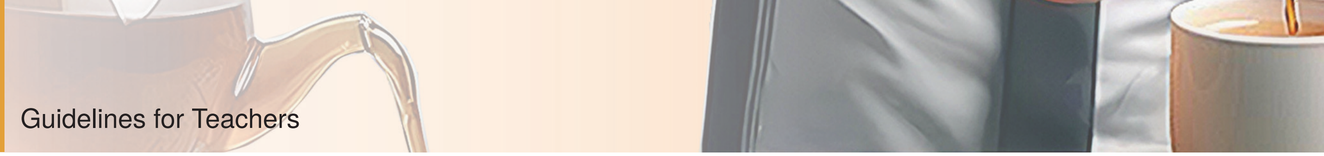
CHAPTER 1

Guidelines for Teachers

Henri Poincaré

Henri Poincaré (April 29, 1854 – July 17, 1912), who was a great French mathematician, philosopher, and Professor at the University of Paris, had a major influence on topology. His work on analysis situs (an old term for topology) introduced basic concepts such as homology and homotopy, which became central to the study of topological spaces. His findings formed a basis for algebraic topology. His contributions constitute a basis for the theory of dynamical systems. Furthermore, Poincaré is known for his elegant writing style. Source: https://en.wikipedia.org/wiki/Henri_Poincaré

PRESENTING a framework for teaching basic concepts of topology to students is the main goal of this chapter. Developing a deep understanding of topology is a process that requires time and patience with teachers playing a crucial role.



1.1 Encouraging students to learn topology

In this section, we offer innovative approaches to get students excited about topology.

1. *Reference to daily life:* To begin, it is helpful to show students how topology is connected to their everyday experiences. For example, you can explain how a donut and a mug (coffee cup) share similar topological properties because they both have a single hole; see Figure 1.1. In addition, you can demonstrate real-life applications of topology in everyday life. Also, talk about how GPS systems utilize topology to find the shortest routes, or how architects use topology in buildings designs. Other possible examples include road maps and directions, the double helix structure of DNA, and railroad tracks and switches.

Figure 1.1. Topologically, a donut and a mug are the same!

Figure 1.2. Note that the hole in the mug handle lines up with the donut hole. Just as we don't pour coffee into the mug handle, we shouldn't pour it into the donut hole! See also the cover.

2. *Topological transformations:* Demonstrate simple topological transformations, such as reshaping play dough to create different shapes, to illustrate how shapes can change while maintaining their essential topological properties.

3. *Engaging visuals:* Use colorful and engaging visuals to illustrate topological concepts. Visual aids like diagrams, images, and videos, can make abstract ideas more tangible and captivating. For example, visual explanations of knots can help clarify complex concepts for students.

4. *Puzzles and games:* Introduce topology through puzzles and games. Maze puzzles, knot puzzles, and games that involve connecting dots or untangling knots can be both interesting and educational. Examples include tangram puzzles, the Tetris video game (joining pieces), jigsaw puzzles (connecting pieces to form a picture), dominoes, and Rubik's cubes.

5. *Incorporate Technology:* Use technology such as interactive simulations and AI-based tools to demonstrate and explore topological concepts. Software can make learning more engaging for young people while searching the internet opens up new worlds for individuals.

6. *Encourage questions:* Create a classroom environment where students feel comfortable asking questions and expressing their curiosity. Encourage them to understand shapes, patterns, and topological transformations.

7. *Introduce renowned mathematicians:* Share stories of influential mathematicians who have made significant contributions to topology, such as Leonhard Euler, Henri Poincaré, and August Möbius. Highlight their achievements and show how they have advanced our comprehension of shapes and spaces.

8. *Field trips and guest speakers:* Organize visits to science museums or invite guest speakers who can share their passion for topology and its practical applications.

9. *Art and creativity:* Inspire students to express their understanding of topology through art and creativity. They can design their own topological shapes or sculptures to display their interpretations of topology.

10. *Group projects:* Assign group projects such that students can explore topological concepts together. For example, they can collaborate to create paper models that highlight the similarities and differences between a loop and a Möbius strip. Collaboration can stimulate creativity and a deeper understanding of the topic.

11. *Problem-solving challenges:* Present students with engaging problems and challenges related to topology. Encourage them to think critically and find creative solutions.

12. *Contests and rewards:* Organize topology-related competitions within the class or school. Rewarding students' achievements, no matter how small, and recognizing their enthusiasm can motivate them to continue exploring the subject.

1.2 Starting activities for teaching topological concepts

Teaching topology can be difficult because of its abstract nature. However, using visual aids and hands-on activities can make learning basic topology more interesting and understandable for young learners. As they continue to learn, they will hopefully be able to explore more complicated topological concepts. In the following activities, we present simple examples from topology.

Activity 1.1

Find the shapes shown in Figure 1.3 and label each shape with the corresponding number.

1. Line segment
2. Crossed line segments
3. Parallel line segments
4. Circle
5. Triangle
6. Rectangle
7. Square
8. Oval
9. Star
10. Diamond
11. Pentagon
12. Hexagon

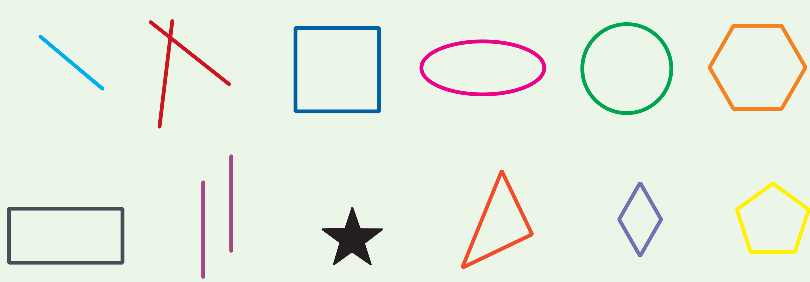


Figure 1.3 Shapes

Activity 1.2

Sketch the capital letters "N, D, C, U, L, O, M, S" and determine whether the end point is the same as the starting point.

A **hole** intuitively means that we can pull a string through it and potentially lift the object.

Activity 1.3

Determine which of the following objects have holes and which do not.

1. Donut	**2.** Straw	**3.** Plate	**4.** Pretzel Bagel
5. Mug	**6.** Tennis Ball	**7.** Balloon	**8.** Button

Figure 1.4 Holes

Activity 1.4

Determine which of the following shapes and objects consist entirely of a single piece:

1. Square
2. Alphabet letter 'i'
3. Tetris L-piece
4. Maze paths
5. Ladder
6. Hopscotch layout for jumping
7. Wifi sign on cellphone
8. Spider web

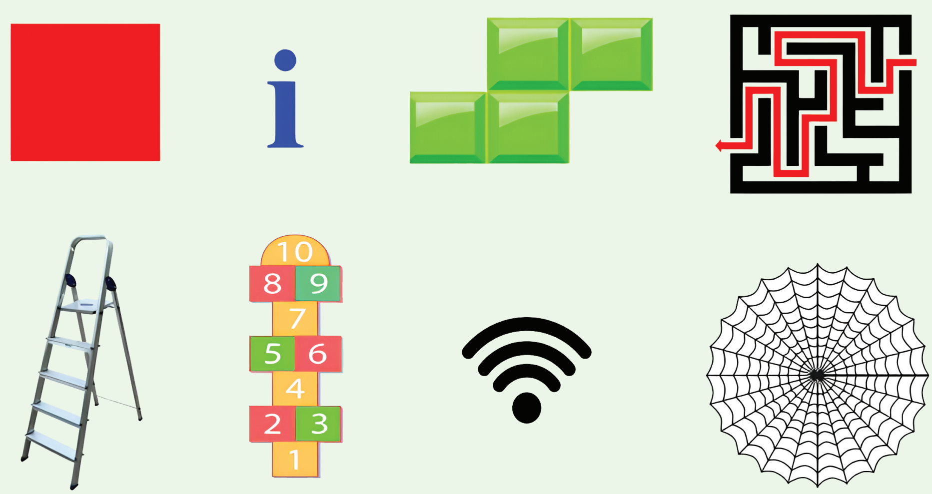


Figure 1.5 Connectedness

Activity 1.5

In addition to the following examples, can you provide other examples of knot handling?

1. Shoelaces tied in a knot
2. Tangled headphone cords
3. Tying a bow
4. Hair tied in a ponytail
5. Tying a ribbon on a gift
6. Knot in a scarf
7. Tied balloon string

Activity 1.6

Which of the following actions include only torsion, stretching, and bending without creating holes, sticking, tearing, or self-intersection? In simpler terms, we are seeking actions that preserve the proximity of points in a shape or object, meaning that they keep close points close and distant points far from each other.

1. Turning a sock inside out
2. Folding a square napkin into a triangle
3. Cutting a pizza into two parts
4. Tearing a piece of bread
5. Inflating a balloon
6. Bending a knee
7. Drawing a rectangle on a tennis ball
8. Drawing a triangle on a balloon and then blowing it up

Activity 1.7

Form a small group with your friends and choose one or two projects from the following list, depending on your preferences:

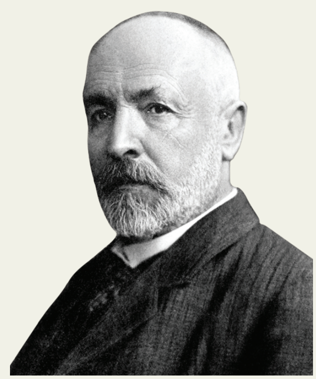
1. Creating a paper chain
2. Designing a treasure map
3. Constructing a 3D puzzle
4. Designing a cardboard house

CHAPTER 2

Intuitive Set Theory

Georg Cantor

Georg Cantor (February 19, 1845 – January 6, 1918) was a renowned German mathematician known for his pioneering work in set theory. He introduced the concept of infinite sets and devised a method for comparing them. Despite facing resistance, he significantly developed our understanding of infinity in mathematics. Cantor's legacy includes the Cantor set and the continuum hypothesis. His work laid the foundation for modern mathematical logic and axiomatic set theory. Source: https://en.wikipedia.org/wiki/Georg_Cantor



Introducing young people to the principles of set theory can be a delightful and interactive endeavor. This chapter offers a straightforward and playful approach to familiarizing students with the elementary concepts of set theory.

2.1 Concept of a set

A collection of various objects (things) is called a **set** and each of these objects is called an **element** of the set. Typically, these objects share a common property. For example, this can include the collection of all numbers less than 10 or the set of all planets in the solar system.

Activity 2.1

Think of different sets, such as a set of fruits in a store or a set of shapes in geometry.

To represent a set, we place its elements inside curly brackets. The capital letters $A, B, \cdots$ stand for sets. For example, the set of even numbers less than 10 can be represented as $A = \{2, 4, 6, 8\}$. If an element a belongs to a set A, we write $a \in A$; otherwise, we write $a \notin A$. For example, $2 \in \{1, 2\}$, but $3 \notin \{1, 2\}$.

A Venn diagram is a visual representation of sets. It facilitates problem-solving by using intersecting and non-intersecting circles (or other appropriate shapes like squares) to illustrate relationships between sets. To draw a Venn diagram for a set, we place the elements of the set inside a circle or a similar shape. Any element that is not included in the set is placed outside of this shape; see Figure 2.1.

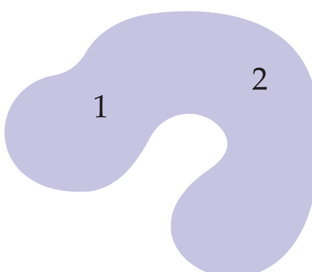


Figure 2.1. Venn diagram for $\{1, 2\}$

A set A is called a **subset** of a set B, denoted by $A \subseteq B$, if every element of A belongs to B. Note that the subset $\subseteq$ is a relation between sets, as shown in Figure 2.2.

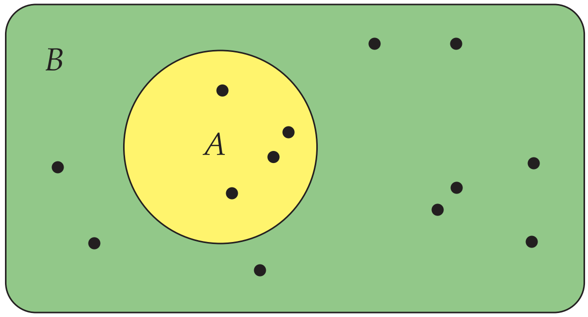


Figure 2.2 Subset relation

Two sets A and B are considered **equal** if both $A \subseteq B$ and $B \subseteq A$, and then we write $A = B$. This means that they have the same elements. For instance, $\{1,2\} = \{2,1\}$. and $\{1,1,2\} = \{1,2\}$, as shown in Figure 2.3. Therefore, if a collection contains duplicate items, we remove the duplicates.

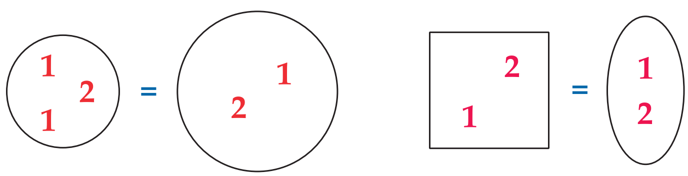


Figure 2.3 Set equality

Activity 2.2

Which of the following relationships are true for $\{1\}$ in Figure 2.4?

1. $1 \in 1$
2. $1 \subseteq 1$
3. $1 \in \{1\}$
4. $1 \subseteq \{1\}$
5. $\{1\} \in 1$
6. $\{1\} \subseteq 1$
7. $\{1\} \subseteq \{1\}$
8. $\{1\} \in \{1\}$
9. $\{1,1\} = \{1\}$

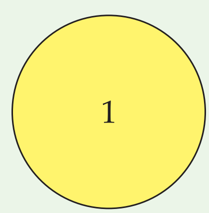


Figure 2.4. Venn diagram for $\{1\}$

Activity 2.3

Which of the following relationships are true for $\{1,\{1\}\}$ and its subset $\{\{1\}\}$ in Figure 2.5?

1. $\{1\} \in \{\{1\}\}$ **2.** $1 \in \{\{1\}\}$

3. $1 \subseteq \{1,\{1\}\}$ **4.** $\{1\} = \{\{1\}\}$

5. $\{1\} \in \{\{1\},1\}$ **6.** $\{1\} \subseteq \{\{1\},1\}$

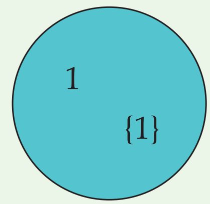


Figure 2.5. Venn diagram for $\{1,\{1\}\}$

Counting the elements in a set allows us to determine the quantity of items within it. Sets with different numbers of elements are distinct from one another.

A set without any elements is called the **empty set** and is denoted by $\varnothing$ or $\{\ \}$, as shown in Figure 2.6, where there is no object inside the circle.

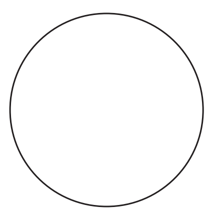


Figure 2.6 Empty set: $\varnothing$

2.2 Set operations

If two sets A and B are given, the set of objects that are contained in both A and B is called the **intersection** of A and B and is denoted by $A \cap B$. The set of objects that are only in A, only in B, or in both is called the **union** of A and B and is denoted by $A \cup B$. Figure 2.7 illustrates the intersection and the union of two sets A and B.

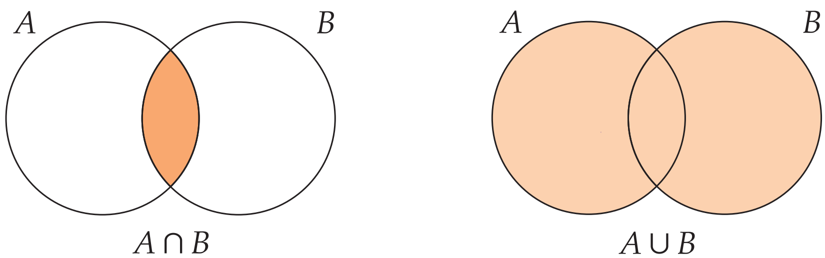


Figure 2.7 Intersection and union of two sets

Example 2.1

The intersection and union of two sets A and B in Figure 2.8 are shown.

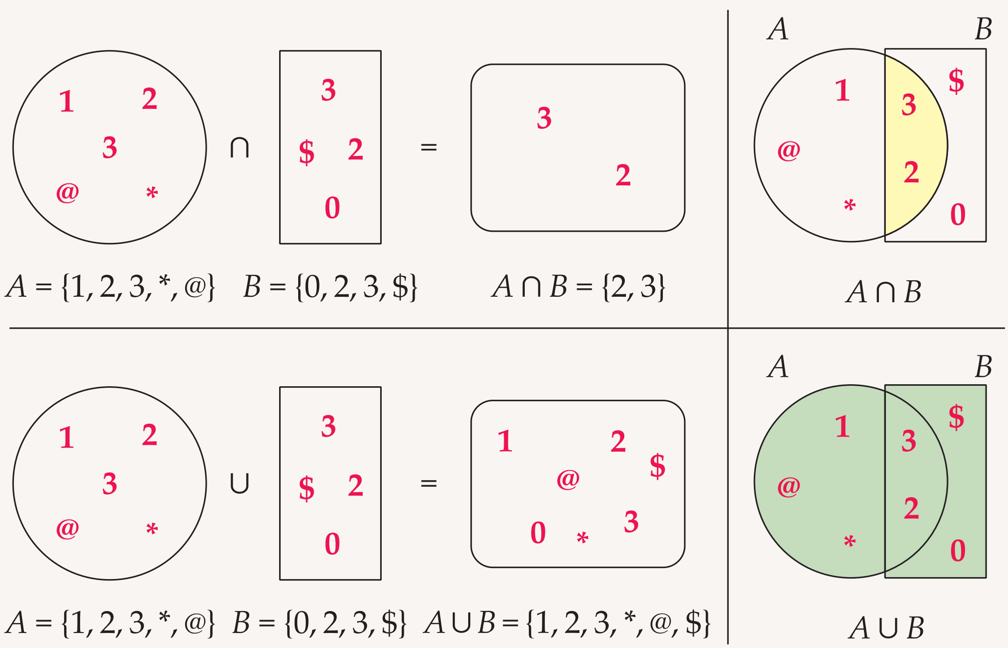


Figure 2.8 Intersection and union

Example 2.2

For three sets A, B, and C we can consider pairwise intersections as well as $A \cap B \cap C$, as shown in Figure 2.9.

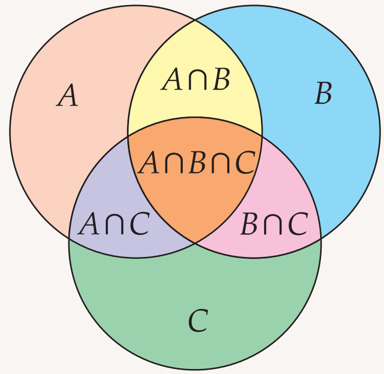


Figure 2.9. Intersection of three sets

Activity 2.4

Hatch the intersection and the union of the pairs of sets in Figure 2.10.

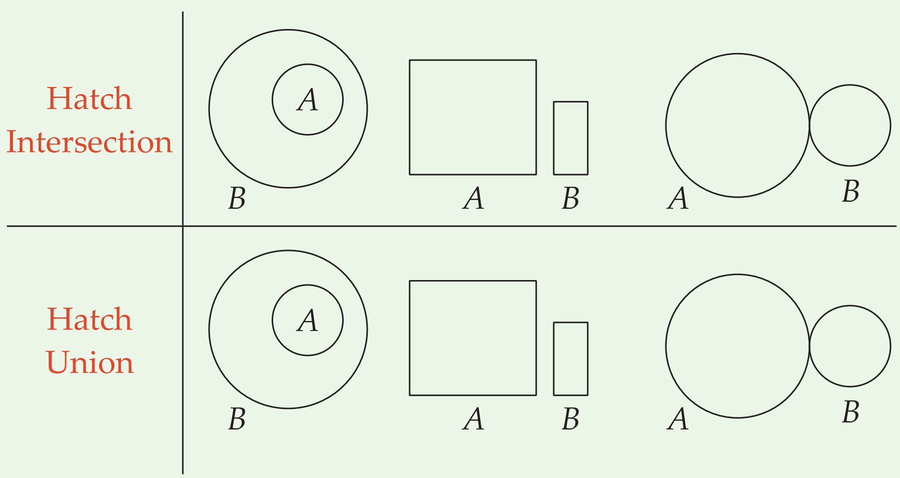


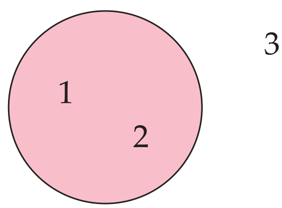
Figure 2.10 Intersection and union of sets

Activity 2.5

Introduce four sets such that the intersection of any two of them is nonempty, but the intersection of all four is empty.

A set that includes all elements dealing with in a conversation is referred to as the **universal set**. Consequently, all sets are subsets of this universal set.

Let M be a universal set and $A \subseteq M$. The set of elements in M that do not belong to A is known as the **complement** of A and is denoted by A^c; refer to Figure 2.11. For two sets A and B, $A - B$ is defined as $A \cap B^c$. Therefore $A - B$ is the set of all elements of A that do not belong to B.



$A = \{1, 2\} \qquad A^C = \{3\}$

Figure 2.11 Complement of a set

Activity 2.6

Hatch $A - B$ with the color green and $B - A$ with the color blue in Figure 2.12.

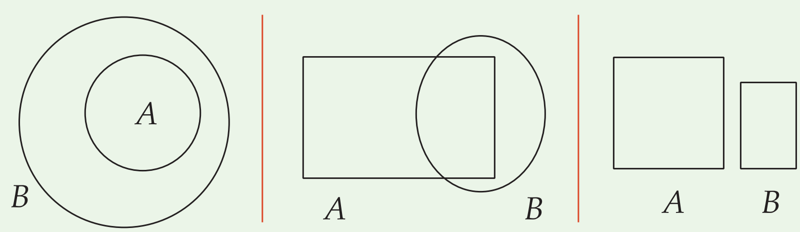


Figure 2.12 Subtraction of two sets

Activity 2.7

For given sets A, B, and C, hatch $A \cap B$, $B \cap C$, and $C \cap A$ in Figure 2.13. Can you find $A \cap B \cap C$.

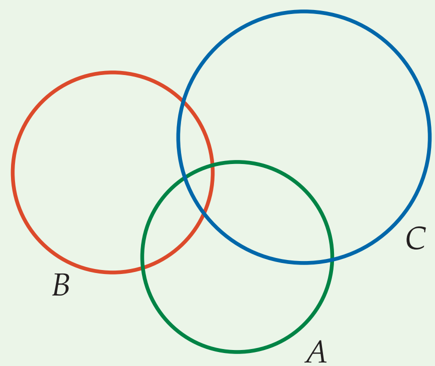


Figure 2.13 Intersection of three sets

Activity 2.8

For given sets A, B, and C, hatch $A \cup B$, $B \cup C$, and $C \cup A$ in Figure 2.14. Can you find $A \cup B \cup C$.

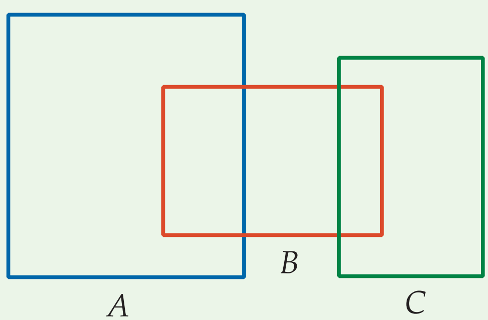


Figure 2.14 Union of three sets

If two sets A and B (which may not be distinct) are given, and we can associate each element of A with only one element of B and vice versa, then we say that there is a **one-to-one correspondence** between A and B, or that A and B are in a one-to-one correspondence.

Example 2.3

The sets A and B are in one-to-one correspondence for all items in Figure 2.15:

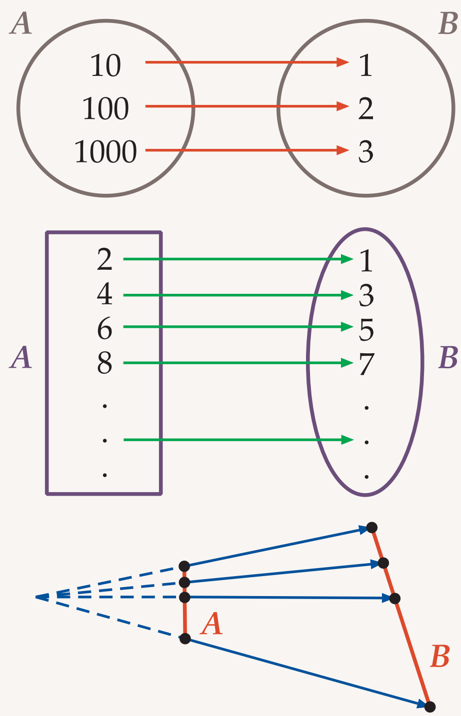


Figure 2.15 One-to-one correspondence between sets

Activity 2.9

Use Venn diagram and show

1. the two sets $\{\clubsuit, \heartsuit, \diamondsuit, \spadesuit\}$ and $\{\star, \pi, 2024, 0\}$ are in a one-to-one correspondence;
2. the sets $\{1, 2, 3, 4, 5, \ldots\}$ and $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ are in a one-to-one correspondence;
3. the set $\{1, 2, \ldots, n\}$ is not in correspondence with the set $\{1, 2, 3, \ldots\}$.

The numbers $1, 2, 3, \ldots$ are called **natural numbers**. We denote the set of all natural numbers by $\mathbb{N} = \{1, 2, 3, \ldots\}$. Any set in the form of $\{1, 2, \ldots, n\}$, where n is a natural number is called a **piece of natural numbers**. Therefore, $\{1\}$ and $\{1, 2, \ldots, 100\}$ are examples of pieces of natural numbers.

A set is said to be **finite** if it is empty or corresponds one-to-one to a piece of natural numbers. Otherwise, the set is called **infinite**.

Given two numbers a and b with $a < b$, the set of all number x such that $a \leq x \leq b$ is called a **closed interval** and is denoted by $[a, b]$. The interval $(a, b) = \{x : a < x < b\}$ is called an open interval. These intervals are all infinite.

Example 2.4

Here, we give examples of finite and infinite sets.

1. The set $\{\clubsuit, \heartsuit, \diamondsuit\}$ and the set of all states of USA are finite sets.
2. The set $\{2, 4, 6, \ldots\}$ and the set of all natural numbers having the digit 0 are infinite.

Activity 2.10

Determine whether the following sets are finite or infinite.

1. $\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\}$
2. The set of animals on the earth
3. The set of kings of the UK
4. The set of all points on a circle
5. The set of edges of a triangular
6. The set of handles of a mug
7. The set of holes on a plate
8. The set of sands on the earth

Activity 2.11

Determine whether the following sets are finite or infinite.

1. The interval $[0, 1]$
2. The interval $(0, 1)$

CHAPTER 3

Points and Curves

Felix Hausdorff

Felix Hausdorff (November 8, 1868 – January 26, 1942) was a German mathematician. He is known for his essential contributions to topology and set theory. He played a key role in developing the concepts of Hausdorff dimension and Hausdorff space. The Hausdorff measure is an important notion in geometric measure theory that laid the foundation for modern topology. Hausdorff, who was Jewish, faced difficulties during the Nazi era, ultimately leading him to take his own life to avoid being sent to the Endenich camp. Source: https://en.wikipedia.org/wiki/Felix_Hausdorff

In this chapter, we introduce fundamental concepts of topology, including curves, neighborhoods, types of points, open and closed sets, boundedness, connectedness, and compactness, all in a simple-to-understand way. A common joke among mathematicians is that a topologist is someone who can't tell the difference between a mug and a donut: Pour coffee into a donut and eat a mug.

A **region** refers to a set of points in the plane, which is a two-dimensional space (see the Section "Four-dimensional space" for notions of the usual two-dimensional and three-dimensional spaces).

A **body** means a set of points in our three-dimensional space. Consequently, a region can also be regarded as a body, as shown in Figure 3.1.

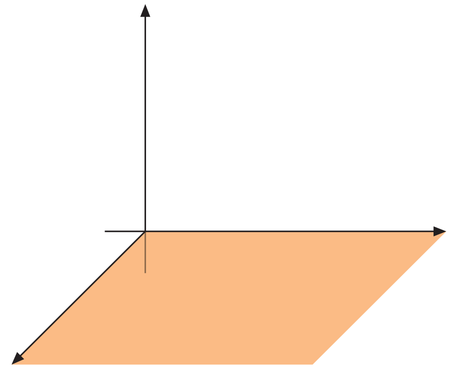


Figure 3.1. A region in the usual three-dimensional space

3.1 Curve

A tiny dot '·' is used in mathematics as a visual representation of a **point**, which indicates an exact position in the plane or space. It has no size or dimensions such as length, width, or height.

Figure 3.2 Point A

To highlight a point, we often represent it as the intersection of two small lines, symbolized as X. We usually designate points with capital letters; see Figure 3.2.

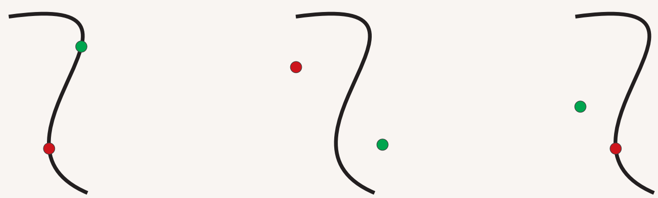
A **curve** is a path that connects one point (the starting point) to another (the end point) without interruption. When you draw a curve with a pencil, the curve should be created without lifting the pencil off the paper and continuing to draw from another point. A real model of a curve is the path we take through a forest from our hut to a river to catch fish. Another example is a map showing overland roads in a country or even a treasure map.

A curve, as a set in the plane, can also be considered a region.

Curves can take the form of either a straight line or a curved line.

Example 3.1

1. Figure 3.3 (i) illustrates two points both of which are on a curve.
2. Figure 3.3 (ii) shows two points none of which are on a curve.
3. Figure 3.3 (iii) presents two points one of which is on a curve and the other is not.



(i) (ii) (iii)

Figure 3.3 Two points and a curve

Activity 3.1

In Figure 3.4, three points are given. Draw

1. a curve through them;
2. a curve such that a point is on the curve and the other two points are not;
3. a curve such that none of the points are on the curve.

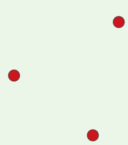


Figure 3.4 3 points

Activity 3.2

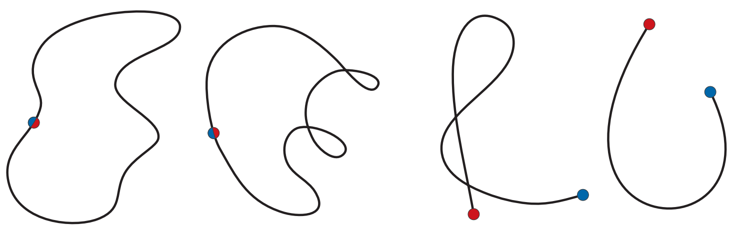
Find a maze on the internet and try to solve it.

3.2 Closed and open curves

A **closed curve** is a curve in which the start and end points coincide. Otherwise, it is said to be an **open curve**; see Figure 3.5. It becomes apparent that a curve can intersect itself at more than one point.

A curve that does not intersect itself (in other words, is non-self-intersecting) is known as a **simple curve**.

A curve can be directed by arrows from its starting point to its end point. For example, curves (a) and (d) in Figure 3.5 are simple.



(a) (b) (c) (d)

Closed Curves Open Curves

Figure 3.5 Closed and open curves

Activity 3.3

1. Is a triangle a simple open curve?
2. Draw a closed curve and an open curve that intersect each other.
3. Draw two closed curves that intersects each other.

The **interior of a closed curve** is the area fully encircled by the curve, while the **exterior of a closed curve** refers to the region beyond the curve, as shown in Figure 3.6.

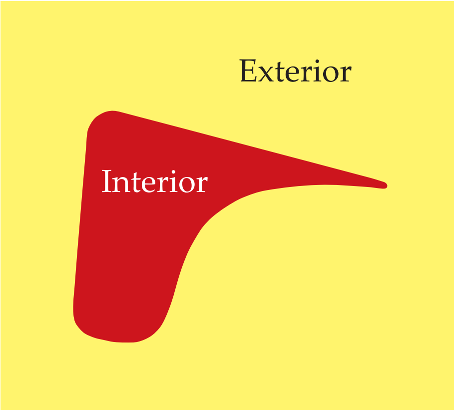


Figure 3.6. Interior and exterior of a closed curve

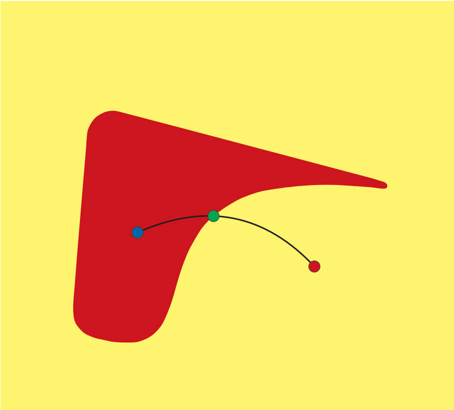


Figure 3.7 Jordan theorem

An interesting result, known as the Jordan curve theorem, which goes back to Camille Jordan (1838–1922), states that every simple closed curve on a plane divides the plane into two distinct regions: an inner and an outer region. Moreover, any curve connecting a point in the interior to a point in the exterior necessarily intersects the curve at some point, as depicted in Figure 3.7.

A simple closed curve is called **oriented clockwise** when we walk on it through its direction, we have its interior on the right side, otherwise, we say that the curve is **oriented counterclockwise**. For example, curve (a) in Figure 3.5 is oriented clockwise.

Activity 3.4

1. Draw a simple closed curve oriented clockwise inside a given circle.
2. Draw a simple closed curve oriented counterclockwise outside a given circle.

Activity 3.5

The points $A, B, C, D, E, F, G, M, N$, and P are given in Figure 3.8. Draw

1. an open curve such that the point A is on the curve;
2. an open curve such that the point A is not on the curve;
3. an open curve through both B and C;
4. a simple closed curve oriented counterclockwise through both D and E;
5. a closed curve such that the point G is on the curve;
6. a simple closed curve such that the point G is in its interior;
7. a closed curve such that the point G is in its exterior;
8. a closed curve featuring the point M in its interior, the point N on the curve, and the point P in its exterior.

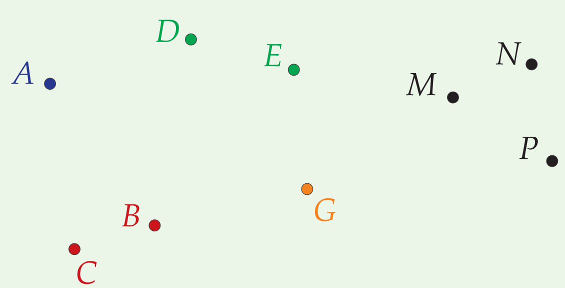


Figure 3.8 Points and drawing curves

Activity 3.6

In Figure 3.9, hatch

1. the interior of the curve with the color red;
2. the exterior of the curve with the color yellow.

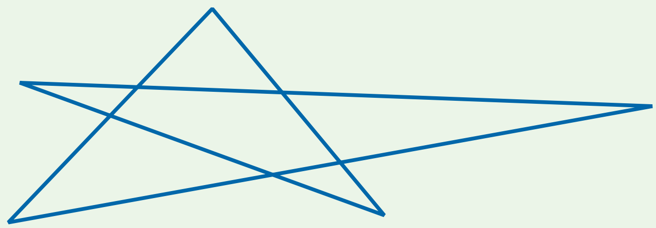


Figure 3.9 Interior and exterior of a curve

Activity 3.7

Figure 3.10. illustrate a closed curve, a point A located in its interior, and a point B situated in its exterior. Draw a curve starting from the point A and ending at the point B. Check to see if this curve intersects the given curve.

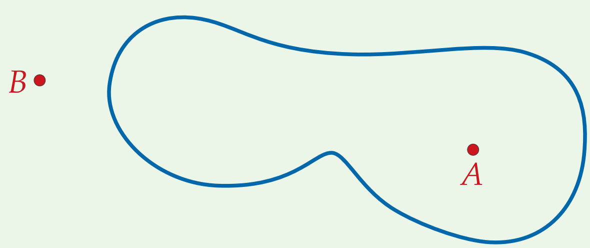


Figure 3.10 Illustrating the Jordan curve theorem

3.3 Interior, exterior, and boundary of regions

A **neighborhood** of a point refers to the interior of a closed curve that surrounds the point. It is important to note that the points on the curve are not considered part of the neighborhood. To clarify this fact, we draw the corresponding curve in the form of a dashed line in the presentation of a neighborhood. It is worth noting that the point itself is included in each of its neighborhoods; see Figure 3.11.

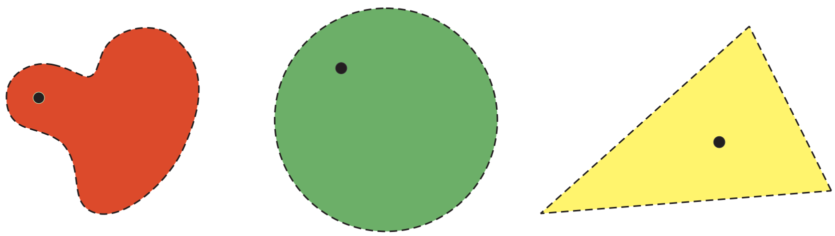


Figure 3.11 Three neighborhoods of three points

Activity 3.8

The points A, B, C, and D are given in Figure 3.12. Draw

1. two arbitrary neighborhoods for A and B;
2. two neighborhoods for C and D such that do not intersect.

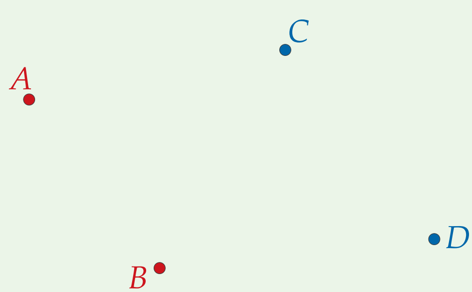


Figure 3.12 Drawing neighborhoods of given points

A point is said to be an **interior point** of a region if there exists a neighborhood that is entirely contained in the region as illustrated in Figure 3.13. An interior point is a part of the region itself. The interior of a region consists of all its interior points.

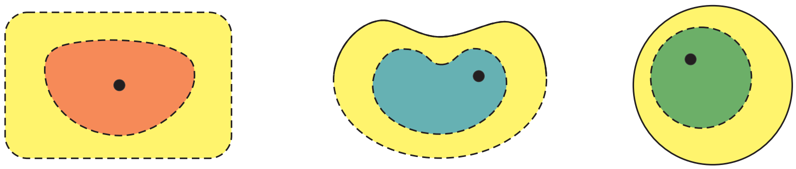


Figure 3.13 Interior points for yellow regions

A point is called an **exterior point** of a region if there exists a neighborhood that is entirely outside of the region, as shown in Figure 3.14. An exterior point of a region is not any part of the region itself. The exterior of a region consists of all its exterior points.

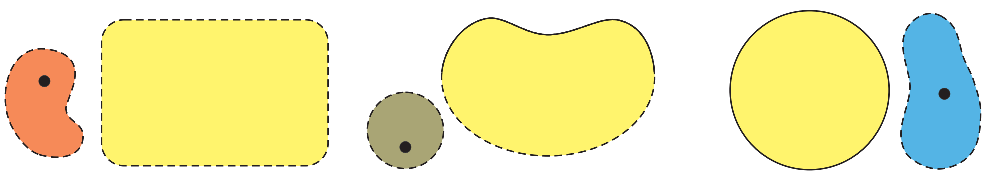


Figure 3.14 Exterior points for yellow regions

A point is considered a topological boundary point, or simply a **boundary point**, of a region if it is neither an interior point nor an exterior point, as shown in Figure 3.15. Therefore, a point is a boundary point if every neighborhood of the point intersects both the region and its complement. A boundary point of a region may belong to the region or not. The boundary of a region consists of all its boundary points.

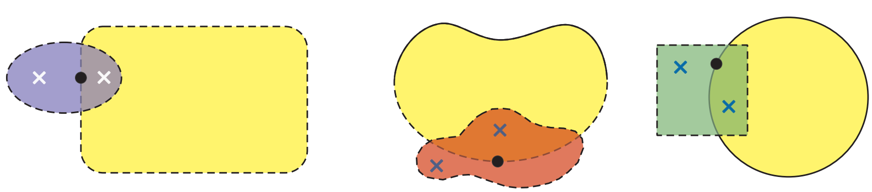


Figure 3.15 Boundary points for yellow regions

Activity 3.9

Find an interior point, an exterior point, and a boundary point for each of the blue regions in Figure 3.16.

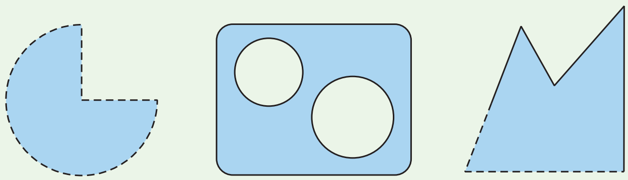


Figure 3.16 Finding specific points

Activity 3.10

Consider the yellow regions in Figure 3.17.

1. Hatch the interior of each of the regions with the color blue.
2. Hatch the exterior of each of the regions with the color green.
3. Color the boundary of each of the regions with the color red.

Figure 3.17 Hatching specific sets

3.4 Open and closed regions

A region is called **open** if it does not contain any boundary points, or in other words, if all its points are interior. It is clear that the plane itself is open. The empty set is also considered open because it does not contain any points.

Activity 3.11

Figure 3.18 illustrates various open regions highlighted in blue. To determine whether each blue region is open, choose any point within the region and draw a neighborhood around it.

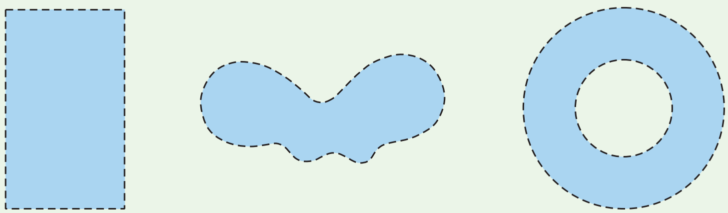


Figure 3.18 Open regions

Activity 3.12

Figure 3.19 shows regions that are not open. To demonstrate this, show that there is no neighborhood around the given point that is completely contained in the red region.

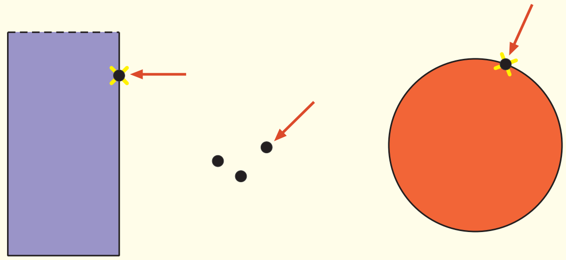


Figure 3.19 Nonopen regions

Activity 3.13

The interior and exterior of a region are always open. Verify this for the orange-colored region in Figure 3.20.

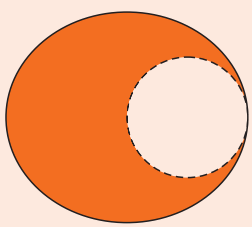


Figure 3.20 Interior and the exterior of a region

A region is called **closed** when it consists of all its boundary points. It is noteworthy that a set that is not closed (or open) is not necessarily open (or closed).

Activity 3.14

Figure 3.21 illustrates several closed regions. Demonstrate that these regions contain their boundaries, which are drawn in a darker color.

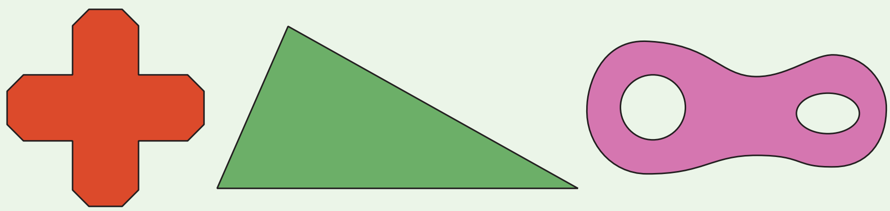


Figure 3.21 Closed regions

Activity 3.15

Figure 3.22 shows some regions (colored in gray, green and red) that are not closed. To demonstrate this, establish that for each region the given point is a boundary point but does not belong to the region.

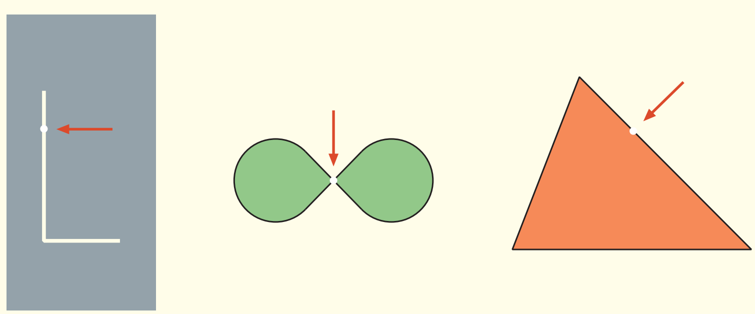


Figure 3.22 Nonclosed regions

It is notable that some regions can be neither closed nor open; see Figure 3.23.

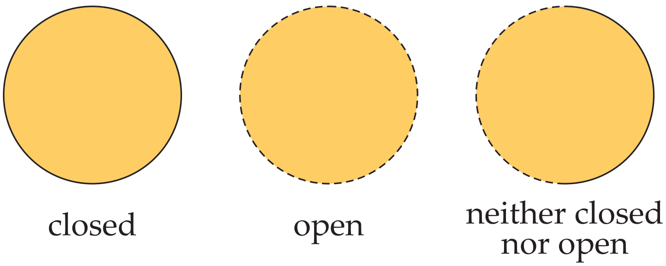


Figure 3.23 A region can be neither open nor closed

The union of any arbitrary number of open regions is open and the intersection of any arbitrary number of closed regions is closed.

For a finite number of open sets, their intersection is also open and the union of a finite number of closed sets is also closed; Figure 3.24 illustrates these facts for two sets.

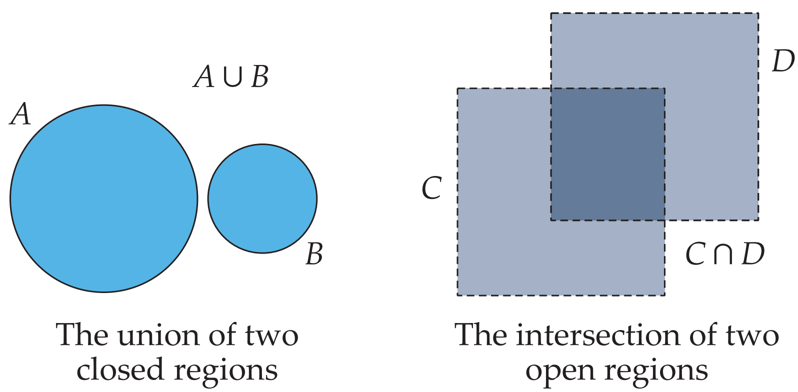


Figure 3.24 Intersection and union of specific regions

Activity 3.16

Which regions in Figure 3.25 are closed, which are open and which are neither closed nor open?

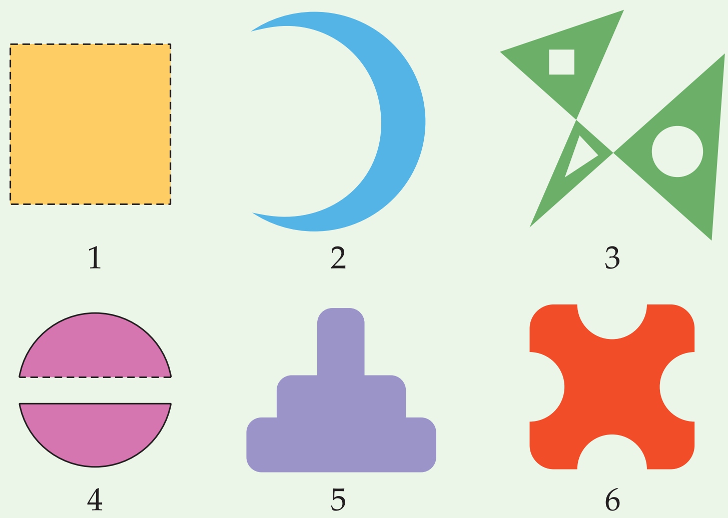


Figure 3.25 Types of regions

Activity 3.17

Provide an example of infinitely many open regions such that their intersection is not open as well as an example of infinitely many closed regions such that their union is not closed.

The set of all open sets in the plane is called the **Euclidean topology on the plane**. The plane equipped with this topology is said to be the **Euclidean two-dimensional space**; For more information, consult Chapter 5 of the book.

3.5 Separation axioms

The separation of points by neighborhoods can be achieved in different ways. In Figure 3.26 we separate two points A and B by using two neighborhoods that do not intersect each other. When it can be done for each two distinct points, we refer to it as the **Hausdorff property**.

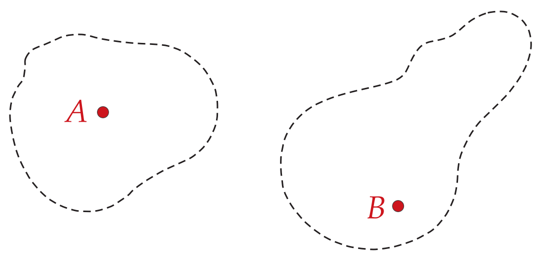


Figure 3.26 Separating two points

Activity 3.18

The points A, B, C, D, E, F, G, and H are given in Figure 3.27. Draw

1. two neighborhoods, one for G and one for H, such that point G is outside the neighborhood of H and point H is outside the neighborhood of G;
2. two neighborhoods, one for each of C and D, such that the neighborhoods intersect;
3. two neighborhoods, one for each of E and F, such that only the boundaries of neighborhoods intersect;
4. two neighborhoods, one for each of G and H, such that the neighborhoods do not intersect.

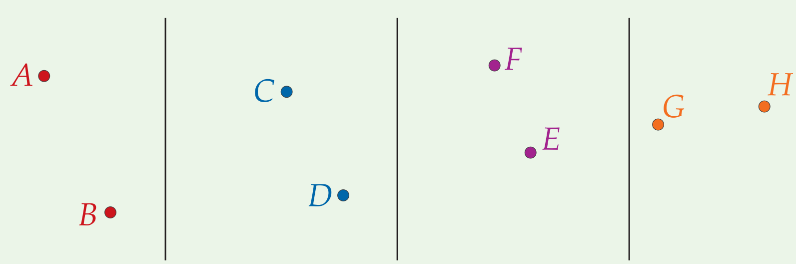


Figure 3.27 Separating points by neighborhoods

3.6 Bounded regions

A region in a plane that lies inside a circle is called **bounded**; Figure 3.28.

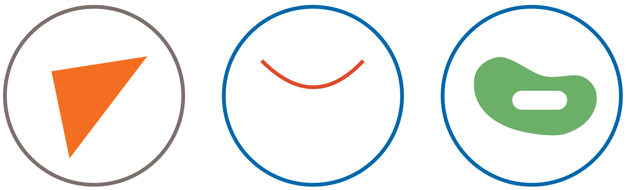


Figure 3.28 Bounded regions

A region is called **unbounded** if it is not bounded.

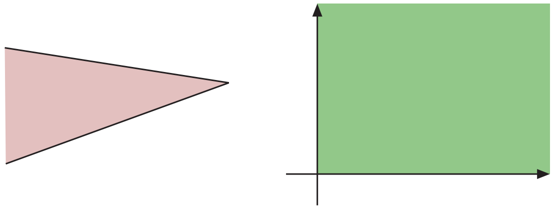


Figure 3.29 Unbounded regions

Activity 3.19

1. Draw a bounded region in the plane.
2. Draw an unbounded region in the plane.

3.7 Connectedness

Connectedness refers to being "all one piece". More precisely, a region is called **connected** if it is not a disjoint union of two open (or two closed) regions. All shapes in Figure 3.30 are connected.

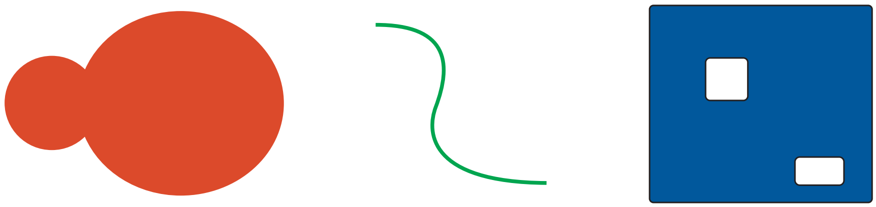


Figure 3.30 Connectedness

A region that is not connected, is called **disconnected**. All shapes in Figure 3.31 are disconnected.

Figure 3.31 Disconnected regions

Activity 3.20

Draw

1. a disconnected region;
2. a connected region.

Activity 3.21

Which shapes in Figure 3.32 are connected and which ones are disconnected?

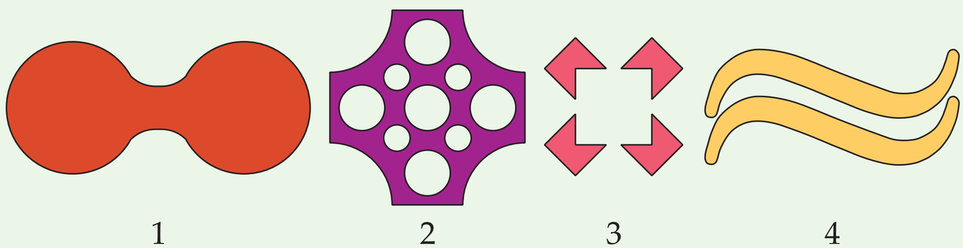


Figure 3.32 Distinguishing connected regions

A connected set within a region is called a **connected component** if it is not contained in any larger connected region. A region may have several connected components. It is evident that a connected region only has one connected component; see Figure 3.33.

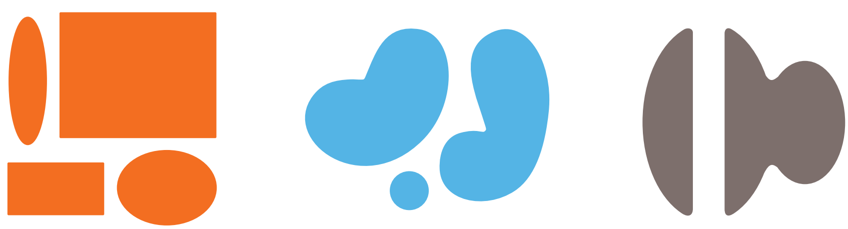


Figure 3.33 Connected components

Activity 3.22

In the plane, draw

1. a region with only one connected component;
2. a region with only two connected components.

3.8 Compactness

A region in the plane is **compact** if it is both bounded and closed[1]. All regions in Figure 3.34 are compact. A region that is not compact is referred to as **noncompact**.

Figure 3.34 Compactness

Activity 3.23

Draw **1.** a compact region; **2.** a noncompact region.

Activity 3.24

Which regions in Figure 3.35, colored in blue, pink, and green, are compact and why?

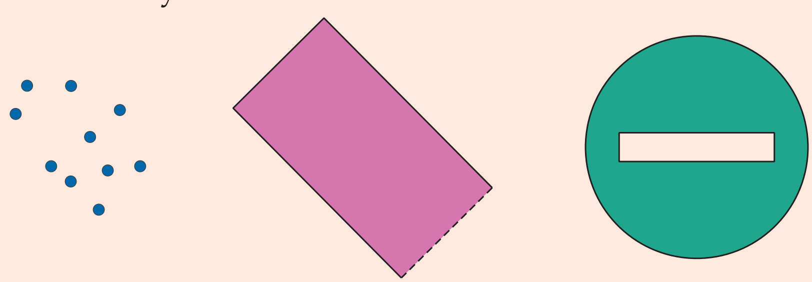


Figure 3.35 Compactness

[1] Alternatively, a set is compact if, for any collection of open sets whose union contains the entire set, there exists a finite number of those open sets whose union also contains the entire set

3.9 Euclidean three-dimensional space

In the plane, as shown in Figure 3.11, we can represent a neighborhood as the interior of a circle. Clearly, the interior of a circle serves as a neighborhood for each of its points.

A **ball** refers to the set of all points on and inside a sphere; see Figure 3.36.

Figure 3.36 Ball

In our three-dimensional space, a neighborhood of a point is defined as the set of all points inside a ball that includes the point itself. Therefore, a point can have many neighborhoods.

The concepts of open, closed, and boundary points of a body can be defined similarly to those in the plane. Accordingly, a body is open if all of its points are interior points. In other words, for every point of the body, we can find a neighborhood (inside a ball) that is entirely contained within the body. The space itself and the empty set are open. On the other hand, a body is closed if its complement (the points outside the body) is open. Also, a body is bounded if it can be enclosed within a ball; see Figure 3.37.

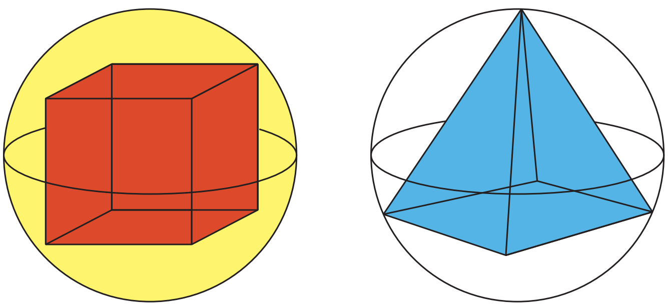


Figure 3.37 Compactness

Otherwise, it is called unbounded; see Figure 3.38.

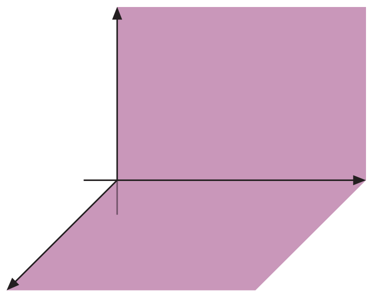


Figure 3.38 Compactness

Similarly, a body is considered connected if it consists of a single piece, and it is said to be compact when it is both bounded and closed.

The set of all open sets in the usual three-dimensional space is called the **Euclidean topology on the space**. The space equipped with this topology is said to be the **Euclidean three-dimensional space**. The abstract definition of topology is provided in the last chapter.

Activity 3.25

1. Introduce some compact bodies around yourself.
2. Introduce some connected bodies around yourself.
3. Introduce some noncompact bodies around yourself.
4. Introduce some disconnected bodies around yourself.

CHAPTER 4

Topological Equivalence and Topological Invariants

Leonhard Euler

Leonhard Euler (April 15, 1707 – September 18, 1783) was a Swiss mathematician and physicist. He is regarded as the greatest mathematician of the 18th century and his influence on mathematics remains unchanged. He made novel contributions to various branches of mathematics such as graph theory, number theory, mathematical analysis, and topology. One of his most elegant achievements is the Euler identity $e^{i\pi} = -1$. Despite losing his sight later in life, he continued to produce an astonishing amount of work. Source: https://en.wikipedia.org/wiki/Leonhard_Euler

This chapter is devoted to the study of topological equivalence, topological invariants, graphs, and Euler characteristics. We also discuss the concepts of holes and handles. It is worth mentioning that topological invariants are employed to demonstrate that two shapes or bodies are not topologically equivalent.

4.1 Topological equivalence

Two mathematical entities are considered to be **topologically equivalent** if they can be transformed into each other without the need to cut or glue. This transformation can involve distortion, stretching, bending, twisting, and shrinking, while avoiding actions such as creating holes, gluing, tearing, and self-intersection. This type of transformation is known as a **topological transformation** or **homeomorphism**. In other words, under a topological transformation, points that are initially close to each other remain close to each other throughout the transformation. They preserve the compactness and connectedness of the entities. In addition, they do not remove any boundaries or add any new ones.

Example 4.1

We can topologically transform a cube to a ball; Figure 4.1.

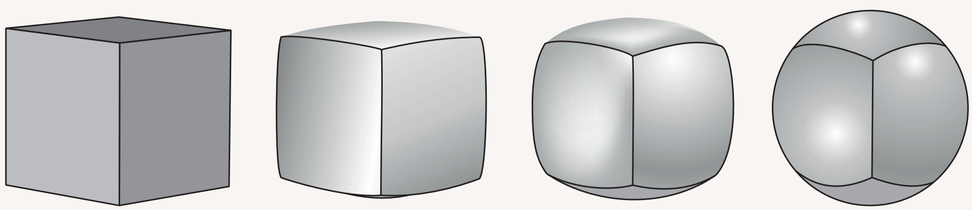


Figure 4.1 Cube and ball

Example 4.2

A donut and a cup have different shapes but share one hole. However, they are equivalent since they can be transformed into each other without cutting or gluing if they are made of clay, as shown in Figure 4.2.

Figure 4.2 Donut and cup

A donut and a plate are not equivalent because a donut has a hole, while a plate does not; see Figure 4.3.

Figure 4.3 Donut and plate

Example 4.3

A square and a triangle are equivalent; see Figure 4.4.

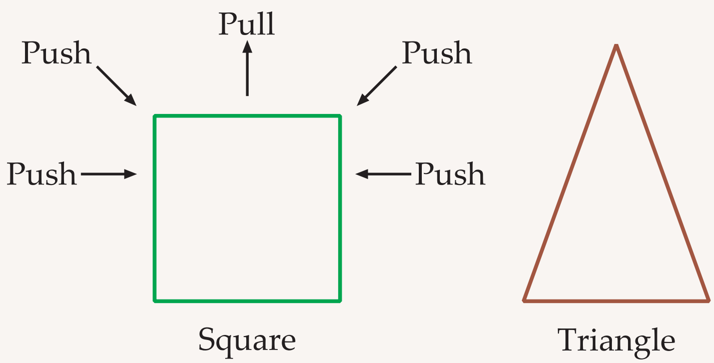


Figure 4.4 Square and triangle are equivalent

Activity 4.1

Demonstrate how a rubber band (Figure 4.5) can be stretched and changed in shape while maintaining its connectedness.

Figure 4.5 Rubber band

Activity 4.2

Take a piece of string with two ends tied together (see Figure 4.6). Lay it on the floor and form a triangle. Then make a rectangle and finally a circle. Explain that triangles, rectangles, and circles are (topologically) equivalent.

Figure 4.6 Piece of string band

Activity 4.3

Use a marker to draw a circle, a triangle, and a square on an uninflated balloon and then inflate the balloon. Explain why the resulting shapes are equivalent to the original shapes; Figure 4.7.

Figure 4.7 Shapes on a balloon

4.2 Topological invariants

A **topological invariant** is a characteristic of a topological space that remains unchanged when it is subjected to a (topological) equivalence. In other words, a topological invariant characterizes the shape or structure of a topological space, regardless of any transformations it may undergo. To establish that two spaces are not (topologically) equivalent, it is sufficient to find a topological invariant that one space possesses and the other does not.

We can immediately observe that neither the “size of objects” nor the “number of corners” are topological invariants. To see this, consider the two shapes shown in Figure 4.8. The two shapes are equivalent despite the circle being larger and having no corners compared to the square’s four corners.

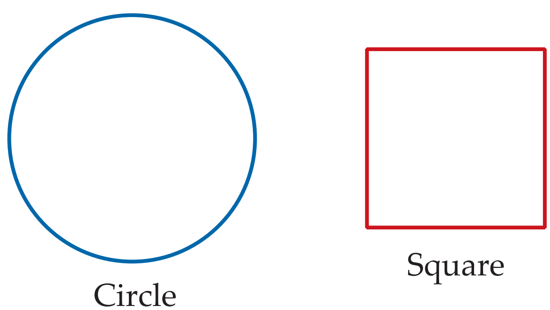


Figure 4.8 Circle and square are equivalent

The most common topological invariants are connectedness and compactness.

In what follows, we explore some other significant topological invariants.

4.3 Component number

The **component number** of a shape in the plane refers to the number of its connected components. This is a topological invariant. If two shapes have different component numbers, then they are not (topologically) equivalent.

Activity 4.4

Use the component numbers to show that the shapes “i” and “u” are not (topologically) equivalent.

Activity 4.5

Use the component numbers to show that the shapes in Figure 4.9 are not (topologically) equivalent.

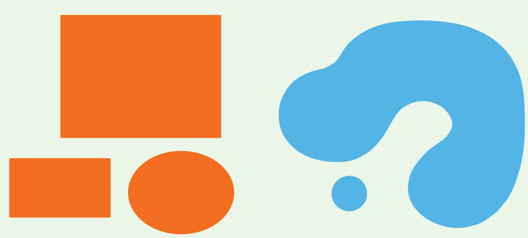


Figure 4.9 Component numbers

4.4 Disconnecting points of curves

A disconnecting point for a curve in the plane is a point that, when removed, changes the component number of the curve. The number of disconnecting points is a topological invariant.

In Figure 4.10, point A is a disconnecting point of the curve, but point B is not.

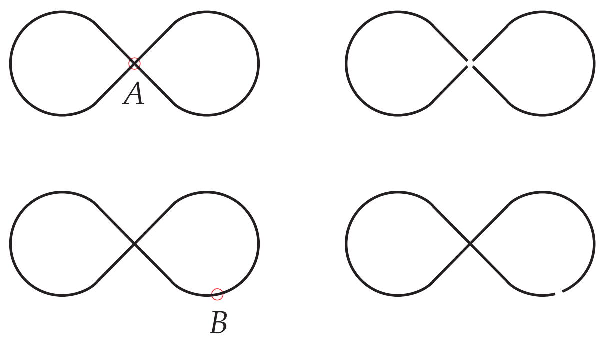


Figure 4.10 Disconnecting point

The curves in Figure 4.11 are not (topologically) equivalent because they have different numbers of disconnecting points.

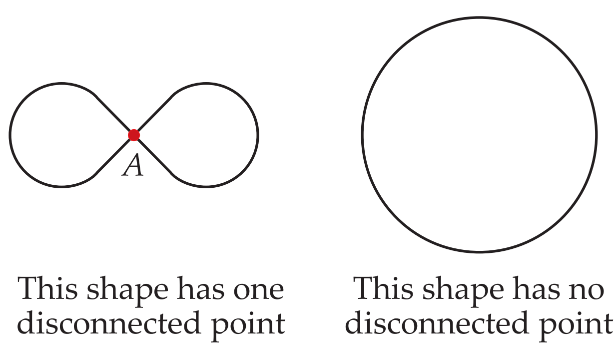


Figure 4.11 Curves with different disconnecting points

A disconnecting point is said to have the **index of disconnecting point** n when removing it changes the component number of the curve to n. The index of a disconnecting point is a topological invariant.
In Figure 4.12, point E has the index 3 while the index of disconnecting point F is 2.

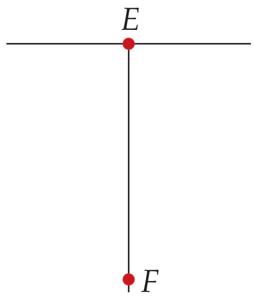


Figure 4.12. Index of disconnecting point

Activity 4.6

Use index of disconnecting points to show that the curves in Figure 4.13 are not (topologically) equivalent.

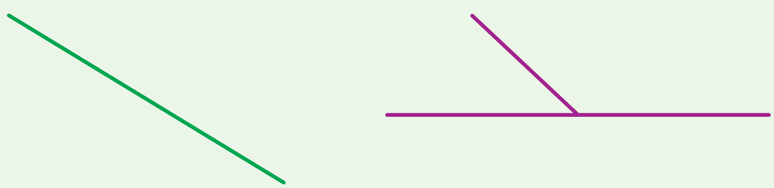


Figure 4.13 Application of disconnecting points

Activity 4.7

Use Figure 4.14 and explain why none of the letters X, Y, and Z are topologically equivalent.

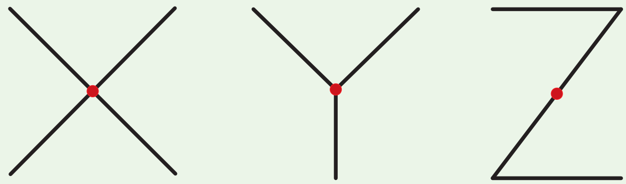


Figure 4.14 Letters X, Y, and Z

Activity 4.8

Use index of disconnecting points to show that the curves in Figure 4.15 are not (topologically) equivalent.

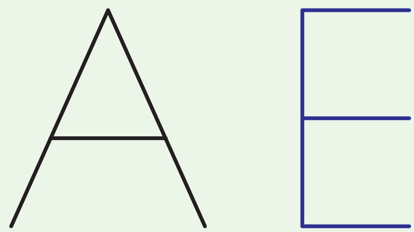


Figure 4.15 Curves having different index of disconnecting point

4.5 Genus number

Intuitively, a **surface** is a connected shape that locally looks like a distorted portion of a plane [1]. Sometimes, it is considered as the boundary of a body in our three-dimensional space. For example, a sphere (the boundary of a ball) is a surface. The torus, the geometric shape of a donut, is also a surface. It can be constructed from a rectangle as follows:

First, bend a rectangle in one direction and connect the opposite sides to form a cylinder. Then, bend the cylinder so that its ends are joined; see Figure 4.16.

[1] More precisely, each of its points has a 'small neighborhood' that topologically equivalent to the interior of a disk

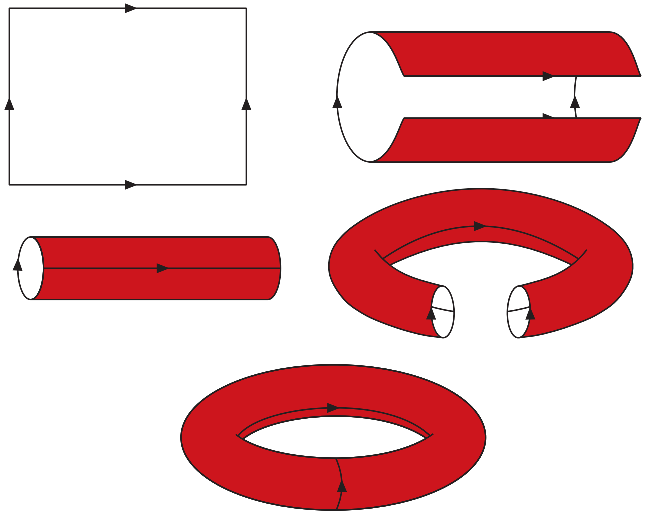
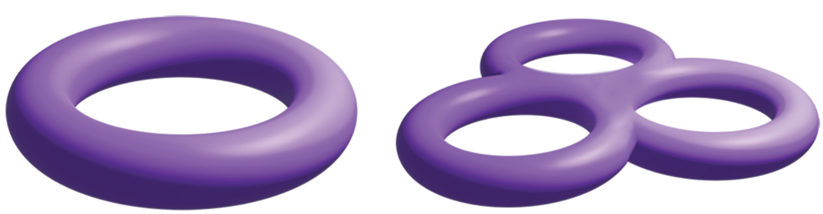


Figure 4.16 Constructing a torus

A **hole** in a mathematical entity refers to a part of it that makes it impossible to continuously shrink the entity to a single point.[2]

The **genus number** of a surface S represents the number of holes it has and is denoted by $\Gamma(S)$. In Figure 4.17, two shapes with different genus numbers are shown.

A sphere has a genus number of 0 while a torus has a genus number of 1. Moreover, a mug, which has a handle, also has the genus number 1. This is why it is often said that "A topologist cannot distinguish between a mug and a donut".



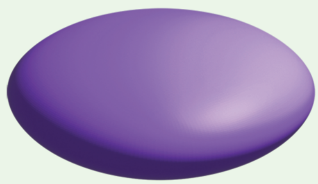
Genus number: **1** Genus number: **3**

Figure 4.17 Genus number

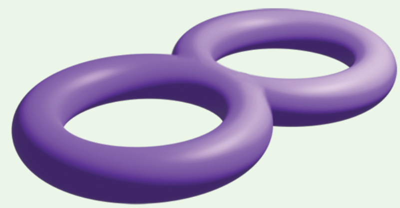
[2]For a rigorous definition of a hole, we need tools such as algebraic topology, specifically, homotopy and homology.

Activity 4.9

Employ the genus numbers to show that the shapes in Figure 4.18 are not (topologically) equivalent.



Genus number: **0**



Genus number: **2**

Figure 4.18 Shapes having different genus numbers

Activity 4.10

Determine the genus number of each shape presented in Figure 4.19.

Brick

Fork

Glass

Ring

Figure 4.19 Genus numbers of shapes

4.6 Winding number

The number of turns of a closed curve in the plane with respect to a certain point is an integer, which is called **winding number**. In other words, this number indicates how many times the curve encircles the point. It is positive if the curve circles the point counterclockwise, otherwise, we represent it by a negative number.

To determine the winding number of a point P shown in Figure 4.20, we draw a ray from P to any point P' on the curve (red line segment PP' in the figure). If we move P' along the curve, the red ray PP' (or the point colored green) rotates around P. Then the number of times the ray PP' (or the point colored green) goes around P is the winding number of P.

In figure 4.20, the winding number of the point P is 3. The winding number for Q is -1 (since the red ray rotates clockwise), and for the point R, it is 0. Curves that have different winding numbers with respect to the same point are not considered equivalent.

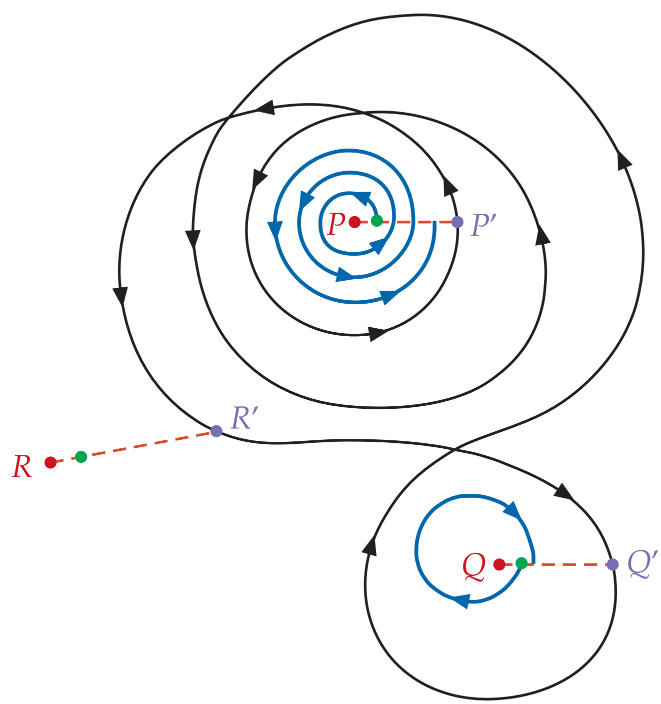


Figure 4.20 Winding number

Activity 4.11

Determine the winding numbers of the closed curve in Figure 4.21 with respect to the points shown.

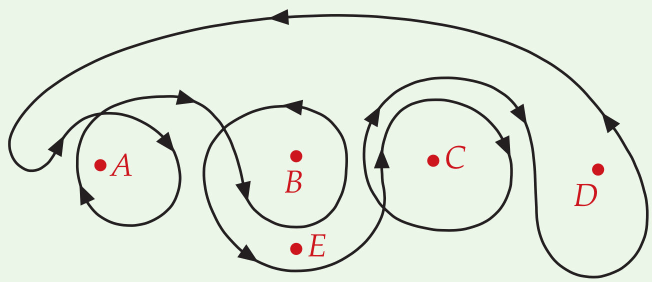


Figure 4.21 Finding winding numbers of A, B, C, and D

Activity 4.12

Can you imagine a closed curve in the plane and a point whose winding number is greater than any number, that is, it is "infinity"?

4.7 Graph theory

The Seven Bridges of Königsberg refers to a historical problem in mathematics, posed by Leonhard Euler in 1735. The problem involves determining whether it is possible to traverse all seven bridges of Königsberg (a city in Russia) exactly once and return to the starting point without retracing any bridge; see Figure 4.22.

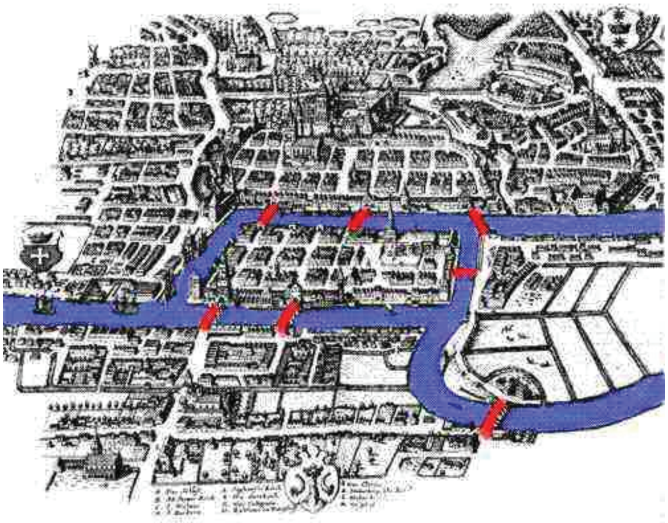


Figure 4.22. Seven Bridges of Königsberg – Source: https://mathshistory.st-andrews.ac.uk/Extras/Konigsberg/

Euler established that such a walk is impossible; Figure 4.23.

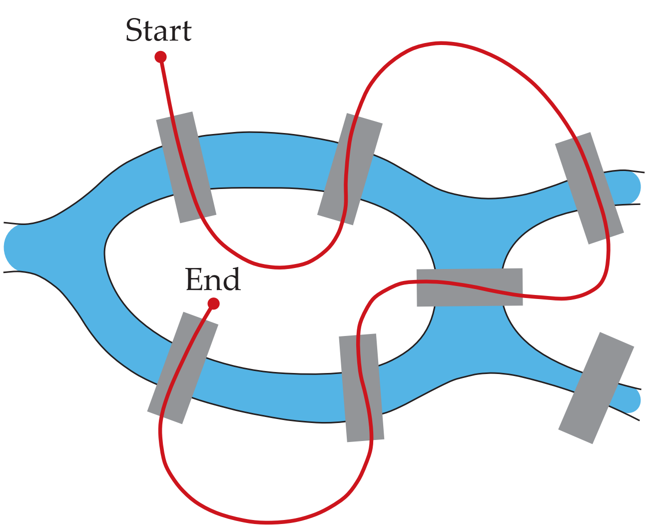


Figure 4.23 Seven Bridges of Königsberg

Euler's groundbreaking work laid the foundation for graph theory and marked a significant development in mathematics. His approach to the

Seven Bridges problem demonstrated the power of abstract thinking and paved the way for the study of networks and connectivity in graph theory. This field has applications in various disciplines, including computer science and telecommunications.

A (finite) **graph** consists of points, called **vertice**s, and the curves connecting them, are called **edge**s. Each edge connects exactly two vertices, and the edges should not cross or intersect unless at a vertex. A square can be considered a graph with 4 vertices and four edges, while a triangle is a graph with three vertices and three edges. These shapes are topologically equivalent even though they do not have the same graph structures.

The edges collectively determine the boundaries of specific regions known as faces. By a **face**, we mean a region between the edges that does not include any edges within it. The outer region is regarded as a face as well. If two faces share common boundary points, they can only share either an edge or a vertex. A **triangle graph** is a graph whose faces are enclosed by three edges. Graphs can be drawn on various surfaces such as a plane, a sphere, or a torus. In a graph, V represents the number of vertices, E stands for the number of edges and F denotes the number of faces.

Activity 4.13

Determine the value of $V - E + F$ for the graph shown in Figure 4.24.

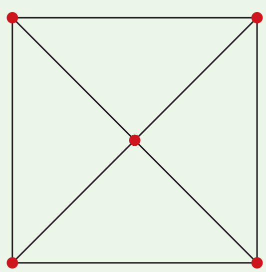


Figure 4.24 Quantity $V - E + F$

A **planar graph** is a type of graph in which its vertices and edges can be drawn on a plane.

A **connected graph** is a graph where every two vertices are linked.

If the set of vertices is finite, the graph is said to be **finite**.

A **map** is a finite, connected and planar graph. **Euler's formula for maps**, which is $V - E + F = 2$, can be computed.

Two questions arise here:

1. Can we draw a map where each of the three houses A, B, and C is connected to all three utilities: water, electricity, and gas?
   The answer is no because when we apply Euler's formula with $V = 6$ and $E = 9$, we get $F = 5$, which is not possible. This nonplanar graph is denoted by G^*.
2. Can we draw a map with five vertices?
   Again, the answer is no because if we apply Euler's formula with $V = 5$ and $E = 10$, we get $F = 7$, which is not possible. This nonplanar graph is denoted by G_*.

Kuratowski's theorem states that a graph G is nonplanar if and only if G contains G^* or G_* as a part (subgraph); see Figure 4.25.

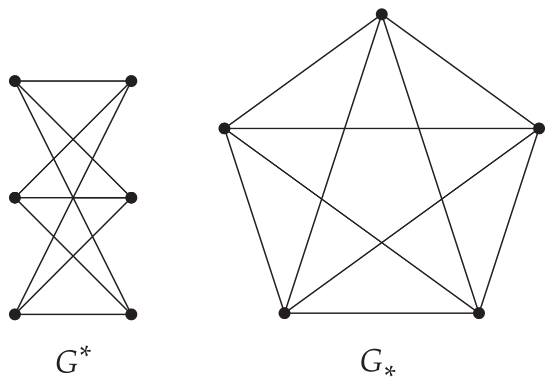


Figure 4.25 Kuratowski's Theorem

A vertex of a graph is said to have a **degree** of n if n edges meet at that point. This degree is a topological invariant in the sense that if one graph, as a shape, has m points each with a degree of n, and the other does not, then the graphs are not (topologically) equivalent.

In Figure 4.26, the point C has a degree of 5, but the degree of D is 3.

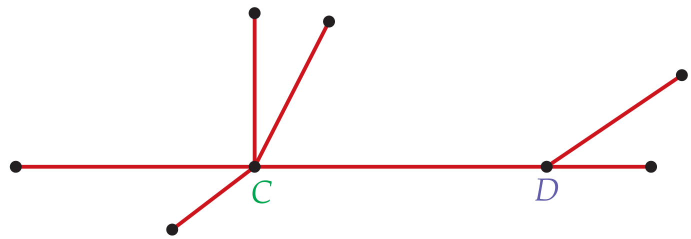


Figure 4.26 Degree of a vertex in a graph

Activity 4.14

Use the degree to show that the shapes in Figure 4.27 are not (topologically) equivalent.

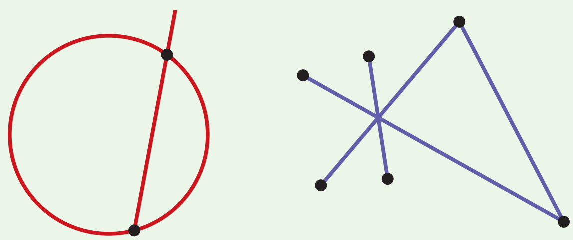
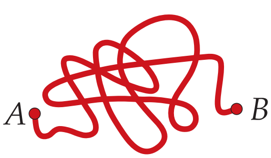


Figure 4.27 Use the degree of vertices



4.8 Möbius strip

Consider a strip of paper measuring, say, 3 cm × 20 cm. If you fold it by bringing the two short edges together and attaching them with glue or tape, you get a loop. Bringing the two short edges together and rotating one of them 180 degrees before sticking it to the other creates a **Möbius strip**; see Figure 4.28.

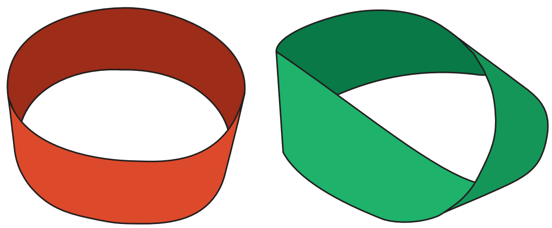


Figure 4.28 Loop and Möbius strip

The Möbius strip is a special type of surface that seems to have two sides, but actually only has one side. If you run your finger along one side, you will end up on the other side without lifting your finger. It's like a never-ending movement!

A **nonorientable surface** is a surface on which traveling along some simple closed curves in a clockwise direction and returning to the starting point results in a change of direction to counterclockwise. Otherwise, it is said to be an **orientable surface**.

To better understand, imagine yourself standing at a point on a large Möbius strip made of glass. On the strip, right beneath your feet, you have written the sentence "I am walking". Begin walking on the surface. When you return to the starting point, you will observe that you are facing the opposite direction and your writing appears as a mirror image; see Figure 4.29. In this case, we say that the surface is non-directional. Surfaces that include a Möbius strip as part of themselves are indeed nonorientable. The remaining surfaces are orientable.

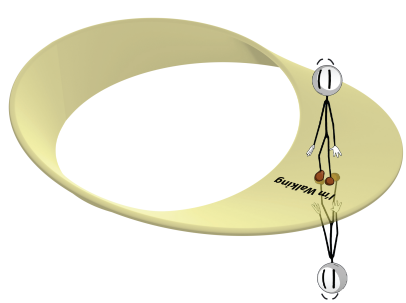


Figure 4.29 Möbius strip is nonorientable

For example, the sphere and torus are orientable while the Möbius strip is not.

Activity 4.15

1. Cut a ring exactly in the middle with scissors. What happens?
2. Cut a Möbius strip in the middle. What happens? Again, cut the resulting strip in the middle with scissors and describe what you get; see Figure 4.30. What happens if you cut it once more?

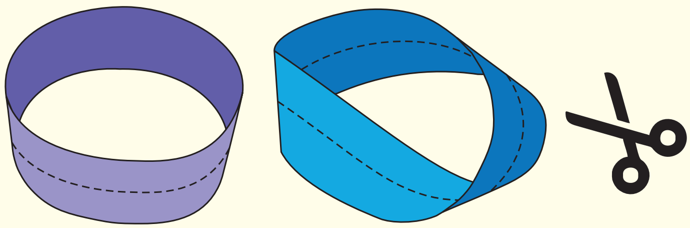


Figure 4.30 Cutting a loop and a Möbius strip

3. Repeat these experiments by making cuts in the ring and the Möbius strip, each with a width of 3 cm, but this time at a distance of one centimeter from the edge.

A point is considered a non-m-boundary point of a surface if every simple closed curve including the point can be contracted to it. Otherwise, it is called an **m-boundary point**[3].

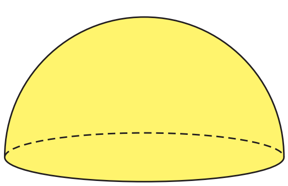


Figure 4.31 m-boundary of a surface

The set of all m-boundary points of a surface is referred to as the **m-boundary** of the surface. To illustrate this notion, let us consider the surface of a hemisphere in our space as an example, as shown in Figure 4.31.

[3]m-boundary refers to manifold boundary. Manifolds are spaces that are locally similar to Euclidean spaces

Some surfaces such as the sphere and the torus do not have an m-boundary. Other surfaces such as a disk (the set of points inside and on a circle) and the Möbius strip have an m-boundary.

A **closed surface** is one that is compact and does not have any m-boundary points. Examples of closed surfaces include the sphere and the torus. Examples of surfaces that are not closed are the Möbius strip and a cylinder.

Activity 4.16

Consider a soccer ball (Figure 4.32) with hexagonal shapes.

Figure 4.32 Soccer ball

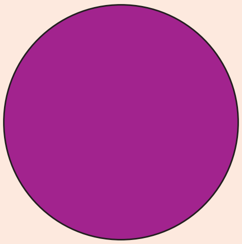


Figure 4.33 Disk

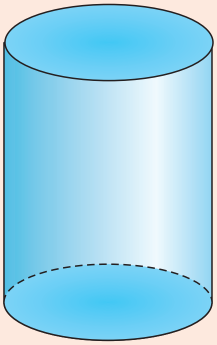


Figure 4.34. Cylinder

1. Make a hole in the soccer ball by removing one of the hexagonal shapes.
2. Show that the punctured soccer ball in step **1.** can be stretched into a disk. Thus, a sphere with a hole is (topologically) equivalent to a disk (Figure 4.33).
3. Explore how a soccer ball with two holes is (topologically) equivalent to a cylinder (Figure 4.34).

4.9 Grandfather Paradox

Do you believe it is possible to travel back in time and influence future events? Possibilities are the essence of science. Let's tell an interesting story related to the so-called grandfather paradox [4]:

Imagine you traveled back in time to your grandfather's birthday, let's say in 1950, and unintentionally killed him at the time of his birth. Since your grandfather died, he could not marry your grandmother, so your mother could not be born in 1980, and consequently, you could not be born in 2010. This creates a paradox where you wouldn't have to go back to 1950 to kill your grandfather. Therefore, your grandfather is alive, your mother is born and you are born as well, as shown in Figure 4.35. [5]

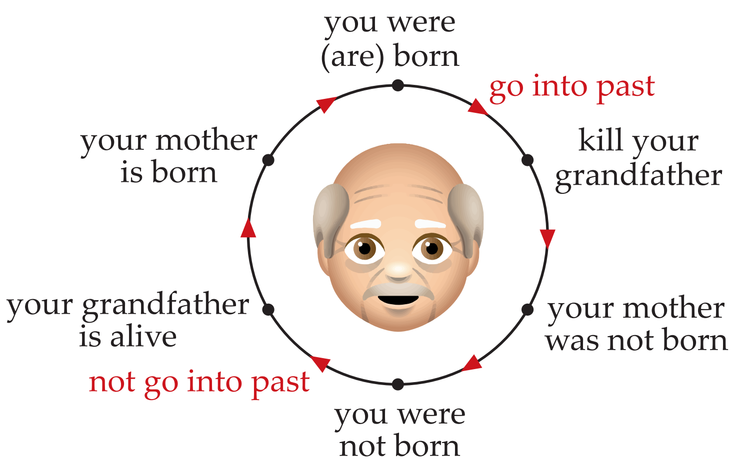


Figure 4.35 The grandfather paradox

One solution is to represent time not on a flat strip, but on a Möbius strip (a twisted loop with only one side). This allows you to move through both "sides" without interruption, as with a loop.

[4] A paradox is a statement that is self-contradictory or contradicts what one would normally expect.

[5] https://www.youtube.com/watch?v=XO1PwyFffBk&t=8s

By plotting the timeline on a Möbius strip, we may avoid the paradox, as shown in Figure 4.36.

Imagine traveling back in time along the Möbius strip and accidentally causing the death of your grandfather on his birthday in 1950. As you travel forward in time, you reach 1980, when your mother could not have been born. Moving forward to 2010, you find yourself in a time when you also could not exist.

By traveling forward along the Möbius strip, once you reach the “back” of the point on the strip corresponding to the day your grandfather was killed, he can be born. If you continue moving, the special properties of the Möbius strip will take you back to your mother’s birthdays and finally to your own.

In this way, we can change events, have contradictory events and return to the beginning time without any problems. By plotting the timeline on a Möbius strip, we can indeed avoid the paradox, as shown in Figure 4.36.

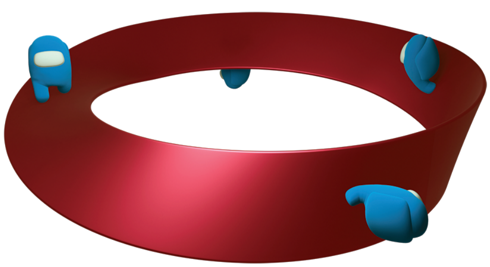


Figure 4.36 Möbius strip and the grandfather paradox

This method is for solving a journey into the past. A journey into the past is only possible if we can change events in the past that do not affect other events in the present and future (relative to this past). Whether it is feasible or not is a question that time itself may one day answer.[6]

[6] https://www.youtube.com/watch?v=JmvHNatZgVI

4.10 Euler characteristic

In this section, we explore **Euler's formula for surfaces**.

A surface is called **triangularizable** if it can be covered by a finite triangle graph. For example, a sphere is triangularizable because we can triangulate it by drawing the equator and a finite number of longitudes, as shown in Figure 4.37.

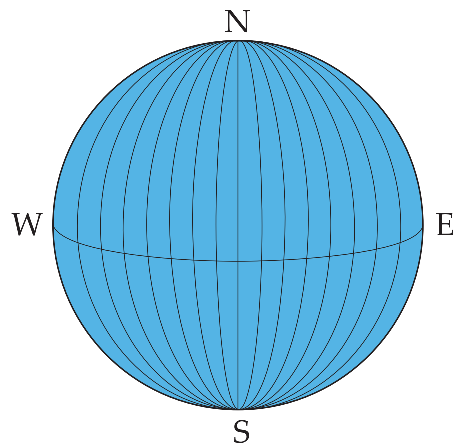


Figure 4.37 Triangularizability of the sphere

A torus is a triangularizable surface. The entire plane is not a triangularizable surface because one cannot triangularize it by a finite triangle graph.

The **Euler characteristic** of a triangularizable surface S is given by $\chi(S) = V - E + F$, where V is the number of vertices, E is the number of edges and F is the number of faces of any covering finite triangle graph. The Euler characteristic remains unchanged regardless of the specific finite triangle graph used. It is a topological invariant. Therefore, in some cases, we can calculate the Euler characteristic of a surface such as a sphere or a torus by considering a topological transformation of the surface, such as a polyhedron; see Figure 4.38.

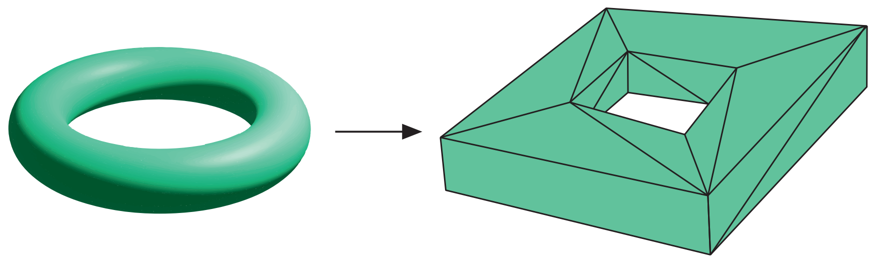


Figure 4.38 Euler characteristic of torus

Recall that a **polyhedron** is a three-dimensional geometric shape made up of flat surfaces (also known as faces) that are polygons. These polygons are connected along their straight edges, and the edges meet at points called vertices. Cubes and pyramids are some examples of polyhedra.

Any Platonic body has an Euler characteristic of 2, as discovered by Leonhard Euler in 1758; Figure 4.39.

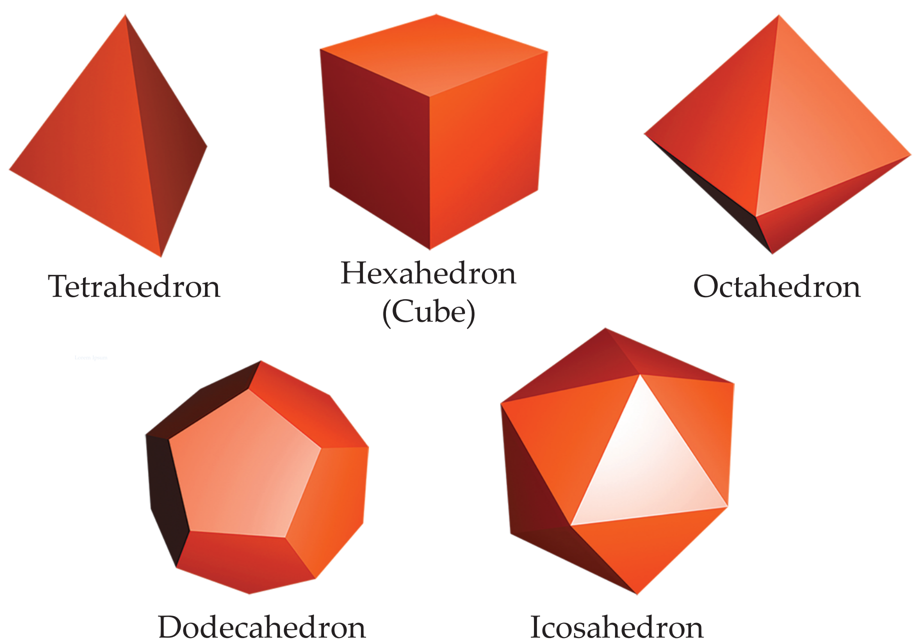


Figure 4.39 Euler characteristic of Platonic bodies

Activity 4.17

Verify Euler's formula ($V - E + F = 2$) for a cube and a pyramid by counting the vertices, edges, and faces in Figure 4.40.

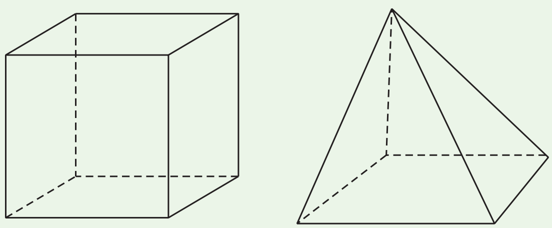


Figure 4.40 Verify Euler's formula for cube and pyramid

Other polyhedra may have different Euler characteristics, as shown in Figure 4.41.

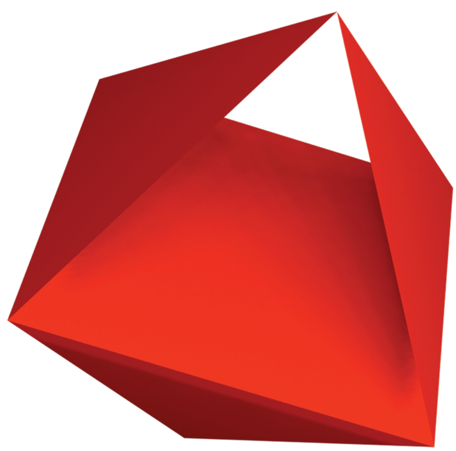


Figure 4.41 Tetrahemihexahedron has 6 vertices, 12 edges, and 7 faces.

The Euler characteristic of a polygon that does not intersect itself, as a surface, is 2; see Figure 4.42.

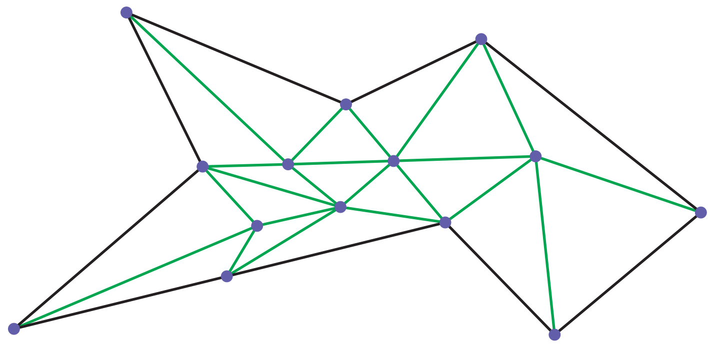


Figure 4.42 The Euler characteristic of a polygon

The Euler characteristic is 2 for a sphere, 0 for a torus, and -1 for a Möbius strip.

Activity 4.18

Find the Euler characteristic of the surface shown in Figure 4.43.

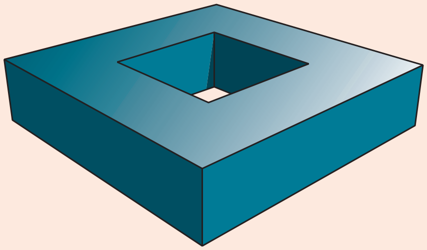


Figure 4.43 Cube with a hole

Activity 4.19

The art gallery problem seeks to determine the minimum number of guards required to monitor the entire gallery. Can you identify any connections between this problem and triangularization?

4.11 Holes and handles

The concept of a hole in a mathematical entity refers to a topological feature that prevents the entity from smoothly contracting to a single point. A classic example is the 'donut-like hole' located in the center of a torus, or a mug with some holes; see Figure 4.44.[7]

Figure 4.44 A cup with three holes!

A torus with a hole has an Euler characteristic of -1 and is called a **handle**.

When two surfaces have 'equivalent' m-boundaries, connecting them[8] creates a new surface with an Euler characteristic equal to the sum of the Euler characteristics of the original surfaces. If we make q holes on a sphere, the resulting surface has an Euler characteristic of $2-q$. If we glue handles to exactly p holes, then the resulting surface has an Euler characteristic of $2-q-p$. In particular, if $p=q$, the resulting surface is denoted by S_p. It is evident that a sphere can be denoted as S_0; see Figure 4.45.

[7] https://www.youtube.com/watch?v=k8Rxep2Mkp8
[8] known as the **connected sum**

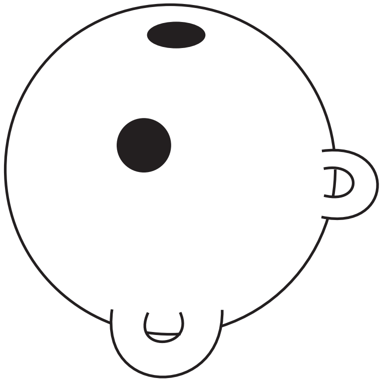


Figure 4.45 A sphere with two holes and two handles

If a sphere has q holes and we glue some Möbius strips onto p of those holes, then the resulting surface has an Euler characteristic of $2 - q$. In particular, if $p = q$, then the resulting surface is denoted by N_p. The surface N_1 is called the **real projective plane**. It is a nonorientable surface that is not topologically equivalent to any body in the Euclidean three-dimensional space.

Activity 4.20

Try to attach a circle to the m-boundary of a Möbius strip!

More precisely, let S be a closed surface. Attaching a handle to S involves removing two open disks (the set of points inside a circle) from S, and then identifying their boundaries. Gluing a Möbius strip to S involves removing an open disk from S and then identifying opposite points on its boundary (circle).

An interesting result of Möbius and Jordan states that every closed surface is (topologically) equivalent to exactly one surface of type S_0 (orientable), S_p (orientable), or N_p (nonorientable).

Furthermore, the Euler characteristic and orientability of a surface can be used to classify closed surfaces. In other words, surfaces with the same Euler characteristic and orientability are (topologically) equivalent.

Now, we can define the genus number accurately: The **genus number** of a closed surface S is $\Gamma(S) = p$ if it is topologically equivalent to S_p $(p \geq 0)$ or N_p $(p \geq 1)$. We point out that the relationship between the Euler characteristics and the genus number for an orientable triangularizable surface S is $\chi(S) = 2 - 2\Gamma(S)$ and for a nonorientable triangularizable surface it is $\chi(S) = 2 - \Gamma(S)$; see [13, page 30].

If we glue two Möbius strips along their m-boundaries we arrive at the **Klein bottle**; see Figure 4.46. The Klein bottle has no m-boundary and is a non-orientable surface. If we place an ant on it, the ant can move inside and outside without hitting any borders. Despite its name, no liquid can be kept inside the Klein bottle. We can represent a Möbius strip in three-dimensional space while the Klein bottle can only be represented (without self-intersections) in a four-dimensional space.

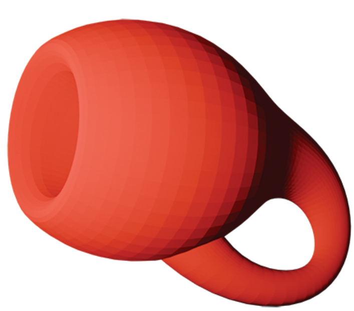


Figure 4.46 Klein bottle

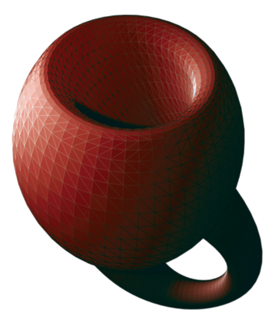


Figure 4.47 Klein bottle

Activity 4.21

Discuss how the Klein bottle differs from ordinary three-dimensional objects.

The Klein bottle has Euler characteristic 0 and is therefore (topologically) equivalent to N_2, as shown in Figure 4.47.

A well-known result, attributed to Hassler Whitney, states that all surfaces can be represented in a four-dimensional space (see Section 5.1).

CHAPTER 5

Challenging Topics in Topology

August Ferdinand Möbius

August Ferdinand Möbius (November 17, 1790 – September 26, 1868) was a German mathematician and astronomer. He was a professor at the University of Leipzig and played a key role in promoting mathematical research in the 19th century. The Möbius strip, which is a non-orientable surface with only one side, carries his name. Möbius made significant contributions to geometry, particularly in the study of projective geometry. Möbius, together with Cayley and Grassmann, was among the first to consider the possibility of geometry in more than three dimensions. Source: https://en.wikipedia.org/wiki/August_Ferdinand_M bius

The topics presented in this chapter may be challenging for high school and first-year university students. The level of difficulty should be adjusted to students' abilities and presented in a way that sparks curiosity and a sence of joy in discovery.

5.1 Four-dimensional space

Teaching students the concept of four-dimensional space can be challenging because our everyday experience is limited to three dimensions. However, one can use simple and creative analogies to help them grasp the basic idea.

A line is one-dimensional, while regions are considered two-dimensional and bodies are regarded as three-dimensional. But what is the fourth dimension?

Some people argue that time is the fourth dimension, as Einstein's theory of relativity incorporates a four-dimensional geometry. Time can be seen as a fourth dimension that allows movement between past and future moments. However, from a topological perspective, there exists a four-dimensional space.

Therefore, we question the significance as we progress through the list of geometric dimensions. In the dimension of 2, we encounter common shapes like circles and squares. These shapes then transform into spheres and cubes in three dimensions. Can we extend the argument and discuss hyperspheres or hypercubes (also known as Tesseracts)?

We can reach a cube from one point in three stages:

In the first step, we connect two points that are one centimeter apart to form a line segment, which is a one-dimensional shape. This is because a small entity, such as a particle, can only move along the forward and backward directions on the line segment.

In the second step, we connect each pair of endpoints of two one-centimeter-long line segments that are parallel and one centimeter apart to form a square, which is a two-dimensional shape. An ant can only move horizontally and vertically or in combinations thereof, on this square.

In the third step, we connect the corresponding corners of two squares that are parallel to each other. For example, the second square is one centimeter in front of the first square. This forms a three-dimensional cube. A human can move in three directions: the aforementioned movements on a flat surface, as well as up and down or combinations of these directions.

Thus, a hypercube can be constructed by joining two three-dimensional cubes that are adjacent to each other by one centimeter, as shown in Figure 5.1.

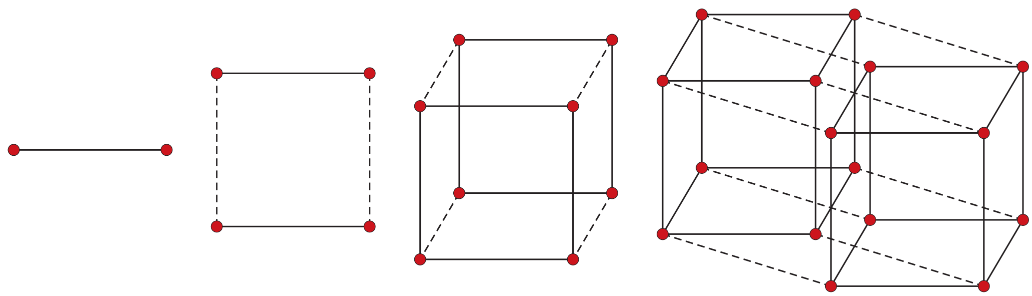


Figure 5.1 Four-dimensional cube

The issue is that we must change direction for each step, moving perpendicular to the previous directions. However, we have already utilized all available directions with horizontal, vertical, upward, and downward movements. As three-dimensional beings, we are unable to transition into the fourth dimension. Our only option is to envision it, ensuring our imagination remains logical and coherent.

Of course, from a physical standpoint, this hypercube is purely imaginary. When we inquire about the number of vertices a hypercube could have, we are essentially asking how many vertices it would possess if it actually existed. This is similar to the classic humorous question: “If you had a brother, would he like kingfish?” The distinction is that inquiring about a non-existent brother is foolish, whereas inquiring about the vertices of a non-existent cube is not foolish because it has a definitive answer.

If a four-dimensional hypercube were to exist, we could uncover many of its properties. We could determine the number of vertices, edges, and faces it possesses. Since the hypercube is constructed by joining two three-dimensional cubes with 8 vertices each, it must have a total of 16 vertices.

In addition to the edges that the two cubes already have, it would also have new edges connecting each pair of vertices, resulting in a total of 32 edges. With some contemplation, it becomes evident that the hypercube would have 24 square faces and 8 cubic hyperfaces.

The following table illustrates the number of "parts" in a line segment, a square, a cube, and a hypercube. This finding demonstrates that the total of these parts is always a power of three.

Object	Dimension	Vertices	Edges	Faces	Hyperfaces	Hypercube
Point	0	1	0	0	0	0
Line Segment	1	2	1	0	0	0
Square	2	4	4	1	0	0
Cube	3	8	12	6	1	0
Hypercube	4	16	32	24	8	1

Table 5.1 n-dimensional cubes ($0 \leq n \leq 4$)

We can think of a line segment as the projection of a square onto a line, which is one-dimensional. Similarly, a square can be seen as the projection of a cube onto a plane, which is two-dimensional (see Figure 5.2). Therefore, we can imagine a three-dimensional cube as the projection of a hypercube onto the three-dimensional space.

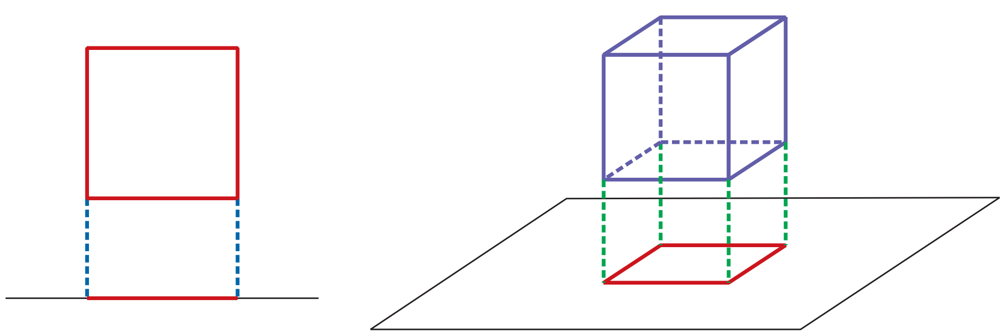


Figure 5.2 Projecting a hypercube

Activity 5.1

A sphere (ball, respectively) is a surface (body, respectively) in three-dimensional space.

1. What is its projection onto the two-dimensional plane?
2. Can you describe a hypersphere and hyperball in four-dimensional space?

To understand the concept of four-dimensional space, we need to approach it from a different perspective. In a two-dimensional space, certain shapes, like the wings of a butterfly, cannot be coincided through transitions and rotations on a flat surface. To properly coincide these shapes, we must move them in three-dimensional space (see Figure 5.3).

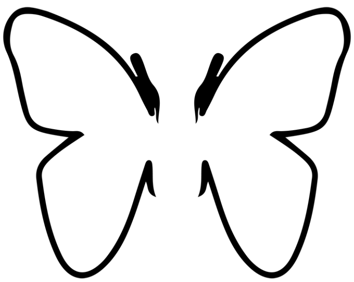


Figure 5.3 Wings of a butterfly

Likewise, in the three-dimensional space that surrounds us, we cannot overlap the left hand with the right hand. To achieve this, we should transfer them from three-dimensional space into four-dimensional space (see Figure 5.4).

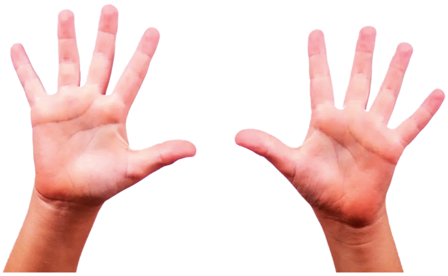


Figure 5.4 Our hands cannot be matched

In this four-dimensional world, we can easily put on a left glove with our right hand; see Figure 5.5.

Figure 5.5 A glove

Figure 5.6 Trapped inside a sphere

Furthermore, if we were trapped in a sphere, we could escape by moving into four-dimensional space.

In 1978, mathematician Thomas Banchoff and computer scientist Charles Strauss at Brown University created computer-generated animations of a hypercube moving inside and outside of our three-dimensional space.

To understand their work, imagine a two-dimensional creature living on the surface of a swimming pool. This creature can only see objects on the surface and is physically limited to two dimensions, just like us in our three-dimensional world. It can only perceive three-dimensional objects through two-dimensional cross-sections. For example, when a cube passes through the water, the two-dimensional creature sees the cross-sections created by the cube cutting through the surface as it enters, passes through, and exits. This idea goes back to the novella “Flatland” by Edwin Abbott and brings to mind the 1980s maze game Pac-Man.

As the cube moves across the surface at different angles and directions, the two-dimensional creature gradually gathers enough information to understand the cube. However, it cannot leave its two-dimensional world.

Strauss and Banchoff’s animations illustrate what we would observe if a hypercube moved through our three-dimensional space at various angles. We see intricate shapes of vertices and edges. Describing these shapes mathematically is one thing, but seeing them in motion is a different and interesting experience.

Consequently, we can gain the ability to comprehend four-dimensional space. In recent decades, there have been significant advancements in visually representing four-dimensional space, and computer science continues to make progress in this area.

5.2 Knot theory

Knots have numerous applications in everyday life, so let's explore their properties. One of the simplest knots is the trefoil knot. To create this knot, we first form a simple loop in a rope and then connect its two ends.

Knots have a rich historical significance dating back to ancient times. According to legend, Gordius, the king of Phrygia, tied an extremely complicated knot that became known as a formidable challenge almost impossible to untangle; see Figure 5.7. Over the years, many tried to untie it, and when Alexander the Great was faced with the task, he was told that whoever could untie the knot would be the ruler of Asia, as local legend dictated. Instead of patiently trying to untangle the rope, Alexander cut it with his sword!

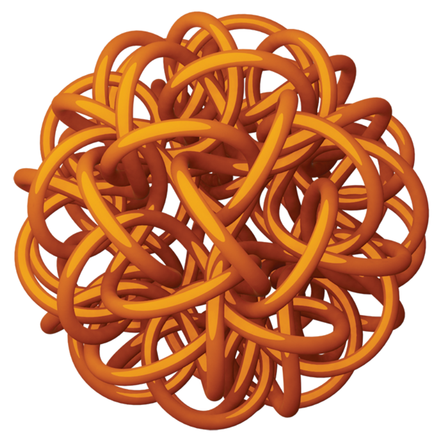


Figure 5.7 Gordian knot

Although a knot is an abstract concept, its influences can be observed in various real-life cases. Textile art often draws inspiration from knot theory, and knot patterns are frequently used in jewelry design. Understanding knot theory can also provide insights into the structure and behavior of DNA molecules. This opens up new possibilities for utilizing DNA to create small structures for medical purposes or for storing information. In addition, knot theory can find applications in the study of robotic arm movements and the planning of their trajectories; see Figure 5.8. [1]

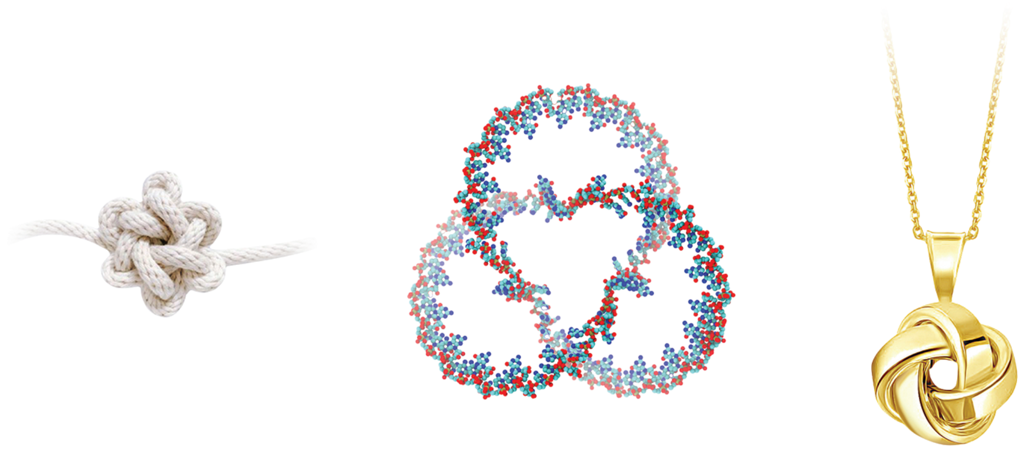


Figure 5.8 Knots

A **knot** is the result of a smooth transition of a circle into the Euclidean three-dimensional space. More precisely, a knot originates from a one-dimensional line segment that is arbitrarily rotated around itself and its two unconnected ends finally join to form a closed loop. It is not topologically possible to continuously convert a knot to a circle. However, we can cut a knot at a point, untie the knot, and finally glue the two points obtained from cutting, to form a circle.

Two knots are said to be **identical (knot-equivalent)** if one knot can be transformed into the other without intersecting or passing through itself. The catalog of non-identical knots exceeds six billion.

The **unknot** (also known as the **trivial knot**) is a closed loop of rope without a tied knot, as shown in Figure 5.9.

The trefoil knot is recognized as the most basic non-trivial knot; see Figure 5.10.

[1] https://www.youtube.com/watch?v=KmhGWCvxKF8

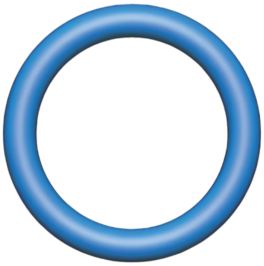


Figure 5.9 Unknot

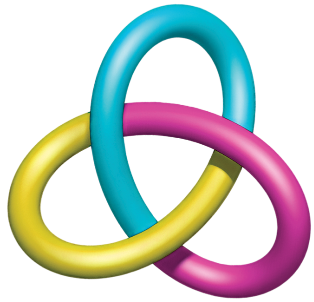


Figure 5.10 Trefoil knot

If we project a knot onto a plane, we obtain a curve with transverse self-intersections called **crossing points**. Such a curve is called a **knot diagram**, in which we distinguish between an overstrand and its substrand at each crossing point. In Figure 5.11, we can see such projections of a knot on three surfaces.

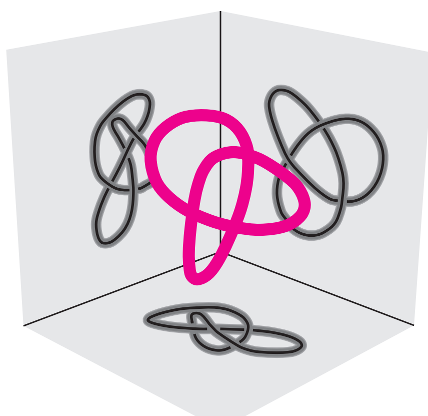
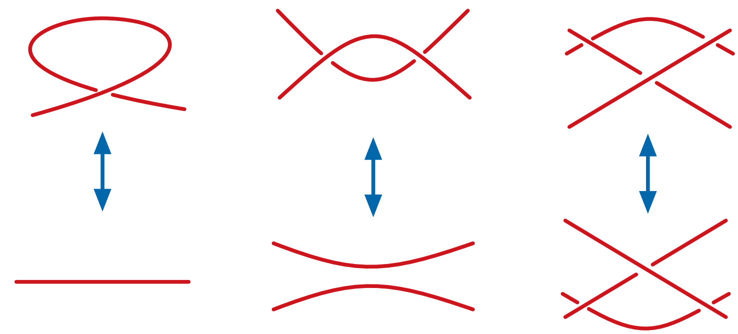


Figure 5.11 Knot diagrams of trefoil

A **knot invariant** is a mathematical quantity or property associated with a mathematical knot that does not change under certain transformations of the knot, such as stretching, bending, or twisting, without cutting or gluing. In other words, it is a characteristic of the knot that remains constant regardless of the specific diagram of the knot. Knot invariants are tools that allow us to prove that two knots are not identical.[2]

Around 1930, Reidemeister introduced three types of moves: (i) Twist move, (ii) Poke move, and (iii) Slide move, as shown in Figure 5.12.



(i) Twist move (ii) Poke move (iii) Slide move

Figure 5.12 Reidemeister moves

These movements, known as **Reidemeister moves**, can be used to distinguish whether a knot is different from another. Reidemeister proved that two knots are identical if their diagrams can be changed into each other through a finite number of Reidemeister moves. Figure 5.13 illustrates how two knots, K_1 and K_2, are identical.

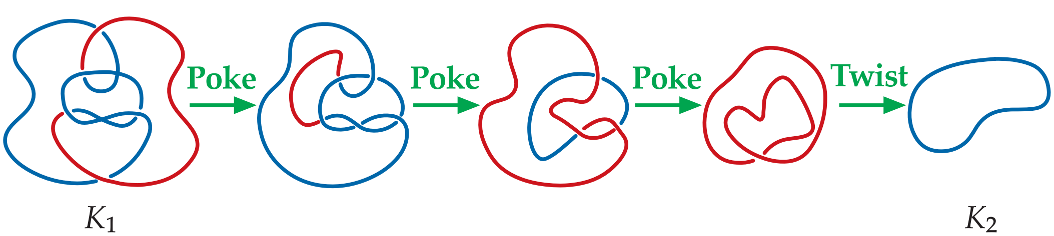


Figure 5.13 Identical knots by Reidemeister moves

[2] https://www.youtube.com/watch?v=8DBhTXM_Br4

Activity 5.2

Use a rope to create the Chefalo Knot, as shown in Figure 5.14. Then demonstrate that it is the unknot.

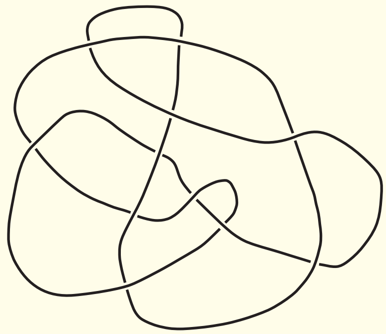


Figure 5.14 Chefalo knot

The **crossing number** of a knot is the minimum number of crossing points in any knot diagram that represents the knot. This number is a knot invariant. For example, the unknot and the infinite-like knot are identical. Their knot diagrams are shown in Figure 5.15. The left curve has no crossing points and the right curve has one crossing point. Thus, the minimum number of crossing points is 0, and therefore the unknot has a crossing number of 0.

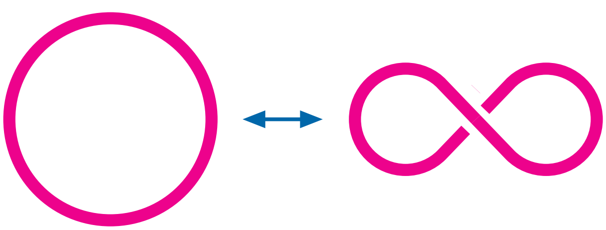


Figure 5.15 Crossing number of unknot is 0

Figure 5.16 shows that there is only one knot (trefoil knot) with a crossing number of 3, one knot (figure-eight knot) with a crossing number of 4, two knots with a crossing number of 5, and three knots with a crossing number of 6. In the symbol n_k shown in Figure 5.16, the number n

stands for the crossing number and k represents the position of this knot in the list.

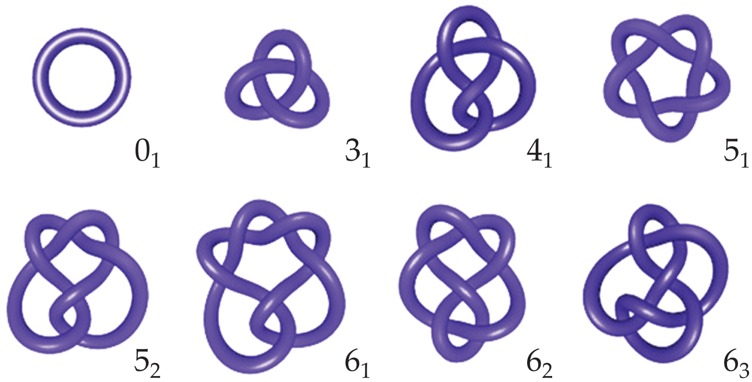


Figure 5.16 Types of knots

We present a simple method for drawing a knot diagram for any given crossing number n[3].

(i) Start by placing n points on the plane as crossings, as shown in the top left figure at 5.17 and assign numbers from 1 to n to each of these points. The number j corresponds to the jth point, which is crossed for the first time.

(ii) Begin drawing the knot from starting point 1 (from the over-strand) and continue in sequential order until reaching crossing n. Lines should not intersect each other.

(iii) After passing through crossing n, proceed to complete the knot by moving to crossing $n-2$ and then to crossing 1 in sequential order.

(iv) Once crossing 1 is completed, pass through either crossing n or $n-1$ (depending on whether n is even or odd) and then through the remaining point ($n-1$ or n), ensuring to pass over one point and under the other.

(v) Finally, close the knot by tying the starting point 1.

[3]This method was presented by Mohammad Ali Gholizadeh in a class project supervised by the author

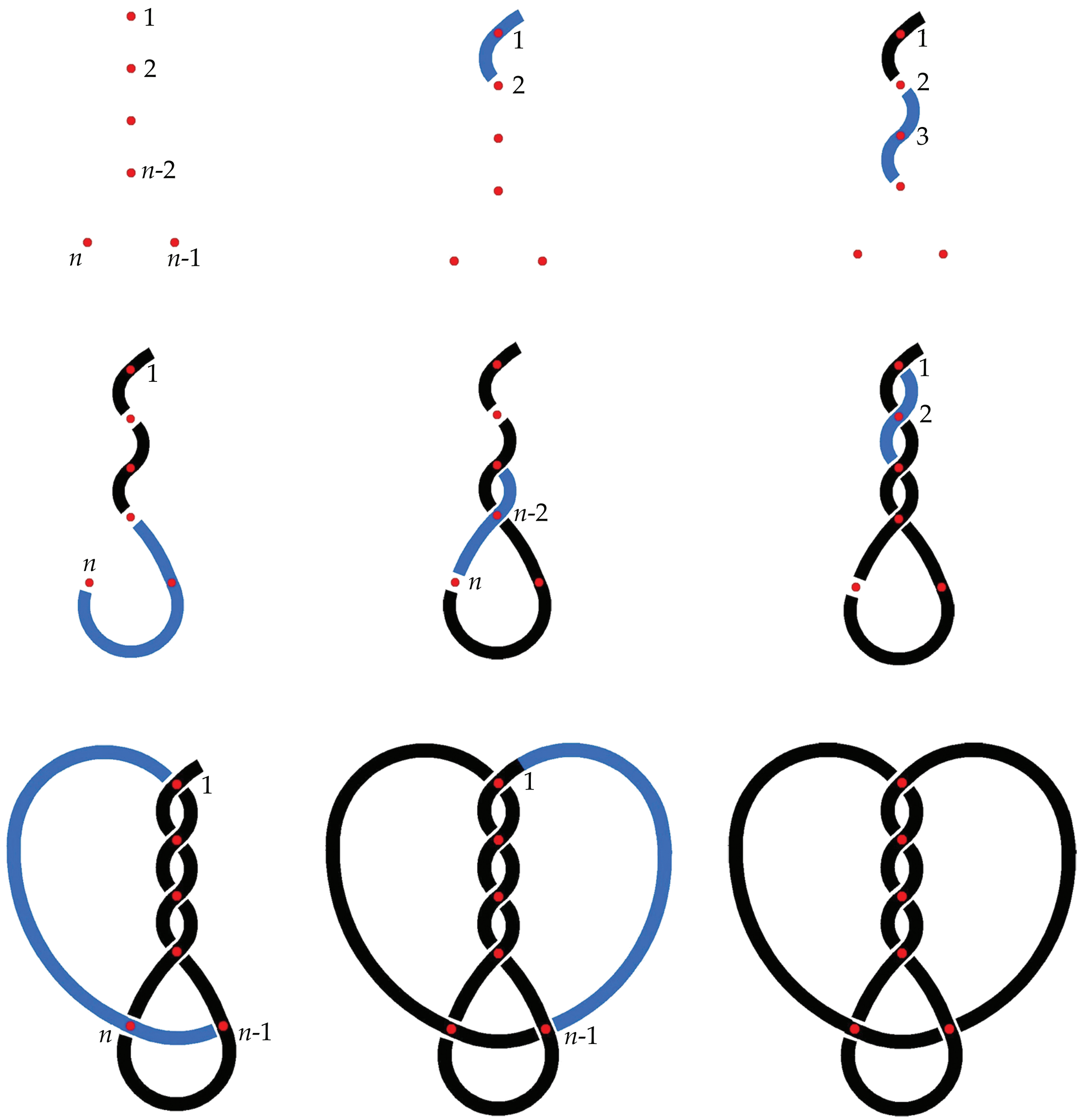


Figure 5.17 Drawing a knot with 6 crossings

When drawing the knot, make sure not to pass through a crossing twice immediately or pass two consecutive crossings in the same manner. In other words, pass over one and under the other. All crossings must be passed both over and under.

An interesting question that can be posed is, "How can we draw all the various knot diagrams of the crossing number n?"

Tricolorability is a knot invariant. A knot is called **tricolorable** if each strand of the knot diagram can be colored with one of three colors such as red, blue, and green, subject to two rules:

(i) at least two colors must be used,

(ii) at each crossing, the three incident strands are either all the same color or all different colors.

For example, the trefoil knot is tricolorable (Figure 5.18), but the figure-eight knot is not (Figure 5.19). If a knot is not tricolorable, we say that it is **nontricolorable**. The unknot is nontricolorable; see Figure 5.20. If a knot is tricolorable, then it is of course not identical to the unknot.[4] [5]

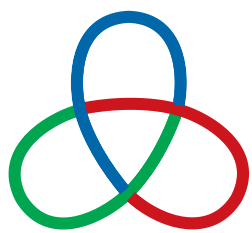


Figure 5.18 Tricolorability oftrefoil knot

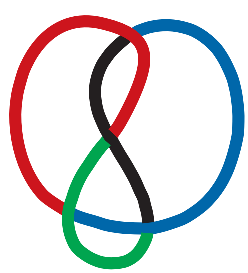


Figure 5.19 Non-tricolorability offigure-eight knot

[4] https://www.youtube.com/watch?v=fwcvmo0y_SI
[5] https://www.youtube.com/watch?v=EBWP1POPc2A

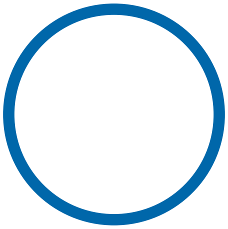


Figure 5.20 Nontricolorability of unknot

Activity 5.3

Take a shoelace, as shown in Figure 5.21, and tie it in different ways to create various knots. Explain how different knots represent topological differences.

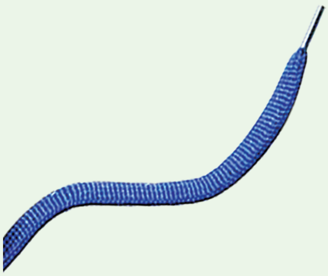


Figure 5.21 Shoelace

Activity 5.4

1. Draw a knot diagram for the granny knot illustrated in Figure 5.22.

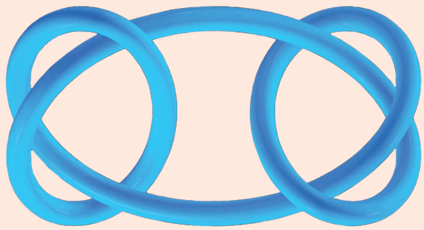


Figure 5.22 Knot diagram of granny knot

2. Use its knot diagram in Figure 5.23 to show that it is tricolorable.

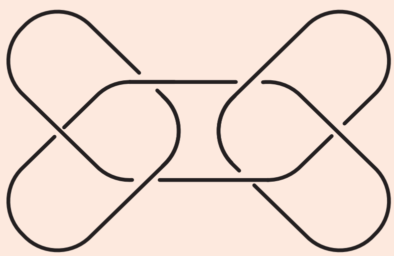


Figure 5.23 Tricolorability of granny knot

Activity 5.5

1. Draw a knot diagram for the square knot illustrated in Figure 5.24.

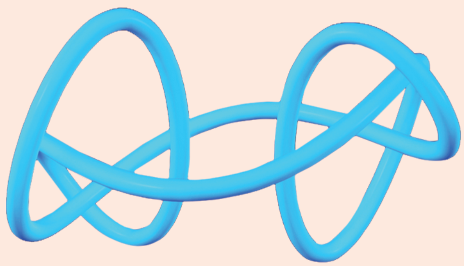


Figure 5.24 Knot diagram of square knot

2. Use its knot diagram in Figure 5.25 to show that it is tricolorable.

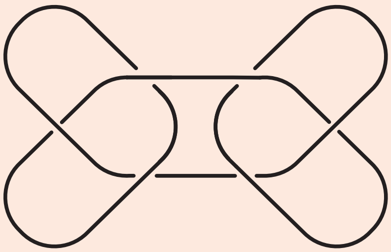


Figure 5.25 Tricolorability of square knot

5.3 Fractal

A **fractal** is a geometric shape that displays self-similarity (endless repetitions), complexity, and detail at all levels of magnification. This means that when we observe a small section of the shape through a microscope, the resulting image resembles the original shape. Fractals can be found in nature such as coastlines, clouds, and trees.

Figure 5.26 presents a fractal[6].

Figure 5.26 Fractal

1. The **Cantor set**, introduced by Georg Cantor, is a fundamental fractal. The construction of this fractal involves an iterative process of removing intervals from a line segment as follows (see Figure 5.27):

   - Start with the closed interval $[0,1]$.
   - Remove the open interval $(1/3, 2/3)$, resulting in two closed intervals: $[0,1/3]$ and $[2/3,1]$.
   - Repeat this process for each remaining closed interval. In other words, remove the open middle third of each interval. After the first iteration, we will have four intervals. After the second iteration, we will have eight intervals, and so on.
   - After an infinite number of iterations, the Cantor set is constructed as the set of points that remain. These points are precisely the ones that were never removed during any step of the construction.

[6] https://drmrehorst.blogspot.com/2018/10/fractal-hand.html

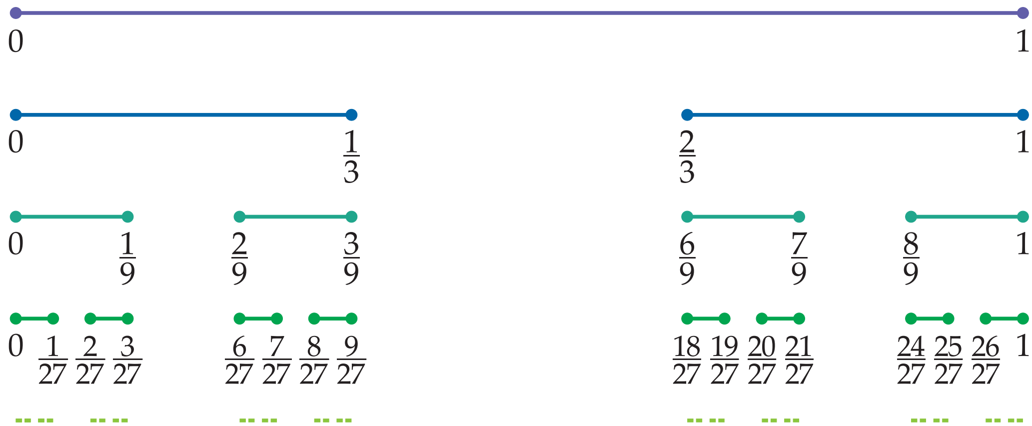


Figure 5.27 Cantor set

Remarkably, the total sum of the lengths of the removed intervals is 1. This set has no interior point and is a disconnected subset of the real line.

Let us recall that a point, a line segment, a square, and a cube are zero-, one-, two-, and three-dimensional, respectively.

We define a set X to have **topological dimension** 0 if for every point in X there exist small neighborhoods whose boundaries do not intersect X; see Figure 5.28. Such a set behaves like a set of isolated points.

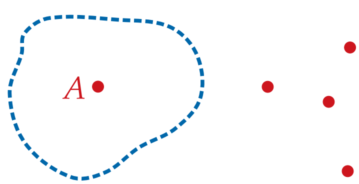


Figure 5.28 A set of topological dimension 0

Activity 5.6

Verify that the Cantor set has topological dimension 0.

2. The **Koch snowflake curve** is created by dividing a straight line into thirds, replacing the middle third with an equilateral triangle, and applying this construction iteratively to each resulting line segment as seen in Figure 5.29.

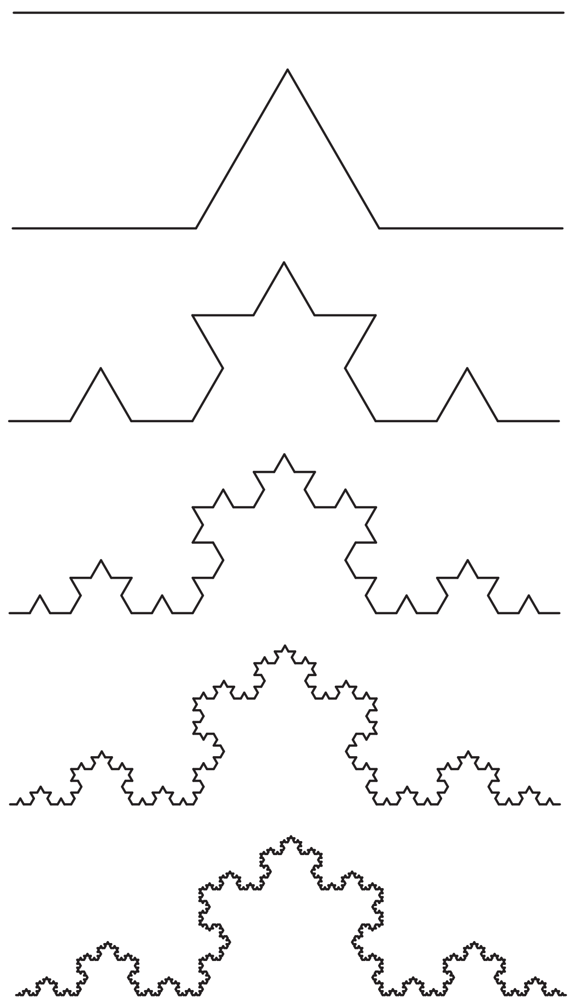


Figure 5.29 Koch snowflake curve

3. The **Sierpiński triangle** (Figure 5.30) is formed by iteratively dividing an equilateral triangle into smaller equilateral triangles and removing the central triangle. This process repeated an infinite number of times, as shown in Figure 5.31.

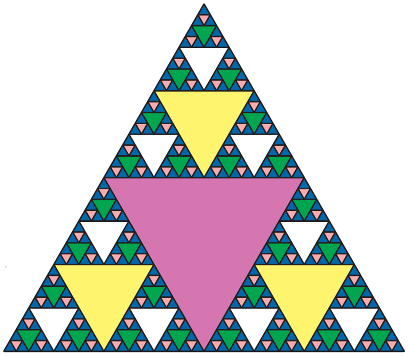


Figure 5.30 Sierpiński triangle

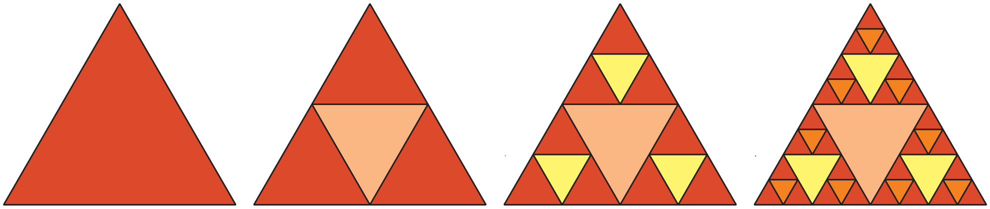


Figure 5.31 Sierpiński triangle

4. The Alexander horn sphere (Figure 5.32) can be constructed using a standard torus by following these steps:

- Begin by cutting out a radial section from the torus.
- Insert a punctured torus on each exposed side of the cut intricately attaching them to the torus on the other side.
- Repeat steps 1 and 2 indefinitely for the two newly added tori.

This topological entity was discovered by James Waddell Alexander II (1888–1971).

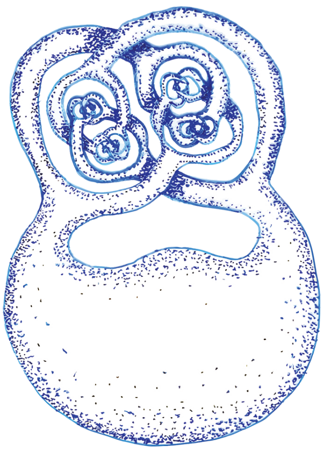


Figure 5.32 Alexander horned sphere

5.4 Abstract topology

In this section, we present an abstract definition of topology and introduce the concept of a topological space. This definition is inspired by Euclidean topologies and is based on the properties of open sets.

Topology

A **topology** $\mathcal{T}$ on a set X is a set consisting of subsets of X such that

(i) $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$;

(ii) The intersection of a finite number of elements of $\mathcal{T}$ is a member of $\mathcal{T}$. In other words, if $F_1, F_2, \dots, F_n \in \mathcal{T}$, then $F_1 \cap F_2 \cap \cdots \cap F_n \in \mathcal{T}$.

(iii) The union of an arbitrary number of elements of $\mathcal{T}$ is a member of $\mathcal{T}$.

When X is equipped with $\mathcal{T}$, it is referred to as a **topological space**.

Example 5.1

One can easily verify that

(1) For any set X, $\{\varnothing, X\}$ is a topology on X, known as the indescrete topology.

(2) For any set X, the set of all subsets of X is a topology on X, named the discrete topology.

(3) $\{\varnothing, \{a\}, \{a, b\}\}$ is a topology on the set $\{a, b\}$; see Figure 5.33.

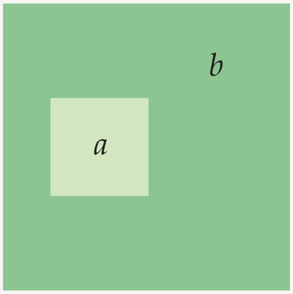


Figure 5.33 $\{\emptyset, \{a\}, \{a, b\}\}$

(4) $\mathcal{T}_1 = \{\{a\}, \{b\}, \{a, b\}\}$ is not a topology on $\{a, b\}$, as $\emptyset \notin \mathcal{T}_1$; see Figure 5.34.

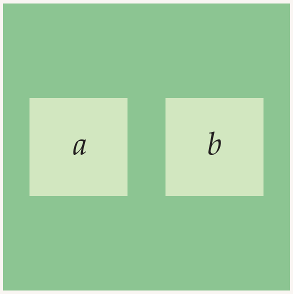


Figure 5.34 $\mathcal{T}_1 = \{\{a\}, \{b\}, \{a, b\}\}$

(4) $\mathcal{T}_2 = \{\emptyset, \{a\}, \{b\}, \{c\}\{a, b, c\}\}$ is not a topology on $\{a, b, c\}$, since $\{a\} \cup \{b\} = \{a, b\} \notin \mathcal{T}_2$; see Figure 5.35.

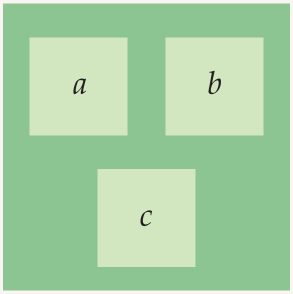


Figure 5.35 $\mathcal{T}_2 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b, c\}\}$

Activity 5.7

Is $\{\varnothing, \{b\}, \{a,b\}\}$ a topology on the set $\{a,b\}$?

Activity 5.8

Is $\{\varnothing, \{1\}, \{2\}\}$ a topology on the set $\{1,2\}$?

Activity 5.9

Is $\{\varnothing, \{10\}, \{100\}, \{10,100\}, \{10,100,1000\}$ a topology on the set $\{10,100,1000\}$?

Activity 5.10

Find all topologies on the set $X = \{a,b,c,d\}$. Can you determine the number of topologies on a set with n elements?

The set of all open sets in the Euclidean topology on the plane or in the space forms a topology, as open sets satisfy all conditions in Definition 5.4. This motivates us to refer to the elements of $\mathcal{T}$ as **open set**s. It is natural that the complements of open sets are called **closed set**s. If $P \in X$ and $G \in \mathcal{T}$ such that $p \in G$, then G is called a **neighborhood** of P. Thus, all notions defined in Chapter 3 can be recovered in the general setting of topological spaces.

An interesting question is whether we can define the Euclidean topology on the line as a one-dimensional space. The answer is yes. To see this, let us consider the line segments without the endpoints on the line. The set consisting of these segments, the unions of any arbitrary number of them as well as the empty set constitutes the Euclidean topology on the line. It is worth recalling that the line itself is the union of all the segments.

CHAPTER 6

Hints and Solutions to Activities

Leonhard Euler

Felix Klein (April 25, 1849 – June 22, 1925) was a German mathematician who made significant contributions to group theory, complex analysis, and geometry. He introduced the Erlangen program, a framework that unified various geometries through symmetry, greatly influencing modern geometry. Klein's exploration of non-Euclidean geometry challenged traditional Euclidean concepts, leading to a deeper understanding of geometry. In 1882, Klein introduced the Klein bottle as a non-orientable surface, expanding the possibilities for mathematical exploration. Source: https://en.wikipedia.org/wiki/Felix_Klein

HINTS and solutions for activities are provided in this chapter. Solving problems is at the core of mathematics. Students are encouraged to try solving the problems independently or together and use this section to check and compare their solutions with the correct answers. Merely reading about problem-solving is not sufficient for mastering it, just like we cannot learn to swim solely from reading swimming instruction books. We must throw ourselves into the water pool to learn to swim. Similarly, we must dive into the sea of thought to tackle problems by understanding the questions, reviewing the literature, and employing various methods. If we find ourselves unable to solve a problem after careful consideration, we can refer to the solutions offered in this chapter. These hints and solutions are designed to guide learners, build confidence, enhance engagement, cater to diverse learning styles, stimulate critical thinking, and facilitate self-paced learning.

6.1 Hints and solutions for Chapter 1

Solution to Activity 1.1

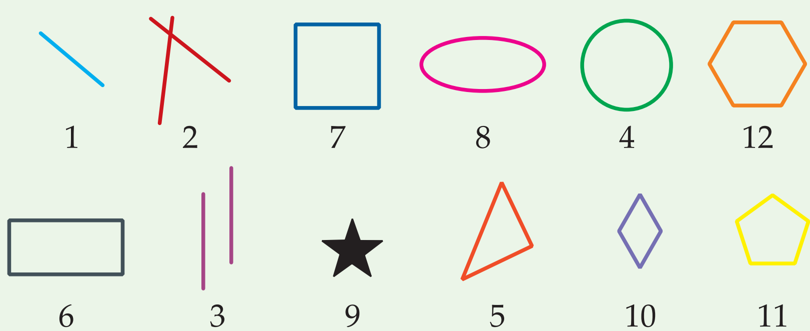


Figure 6.1 Shapes

Solution to Activity 1.2

For each of “D” and “O”, the end point is the same as the starting point.

Solution to Activity 1.3

The objects donut, straw, pretzel bagel, mug, and button all have holes.

Solution to Activity 1.4

Square, Tetris L-piece, maze paths, Hopscotch layout, ladder, and spider web all consist entirely of a single piece.

Solution to Activity 1.5

1. Knots used by surgeons to tie off blood vessels
2. Rock climbing knots
3. Knots used by sailors

Solution to Activity 1.6

The following actions involve only torsion, stretching, and bending.

1. Turning a sock inside out
2. Inflating a balloon
3. Bending a knee
4. Drawing a rectangle on a tennis ball
5. Drawing a triangle on a balloon and then blowing it up

Hint for Activity 1.7

A group of two or three students seems to be enough.

6.2 Hints and solutions for Chapter 2

Hint for Activity 2.1

Share your examples with other students or teachers to ensure they are suitable.

Solution to Activity 2.2

$1 \in \{1\}$, $\{1\} \subseteq \{1\}$, and $\{1,1\} = \{1\}$ holds true.

Solution to Activity 2.3

$\{1\} \in \{\{1\}\}$, $\{1\} \in \{\{1\},1\}$, and $\{1\} \subseteq \{\{1\},1\}$ holds true.

Solution to Activity 2.4

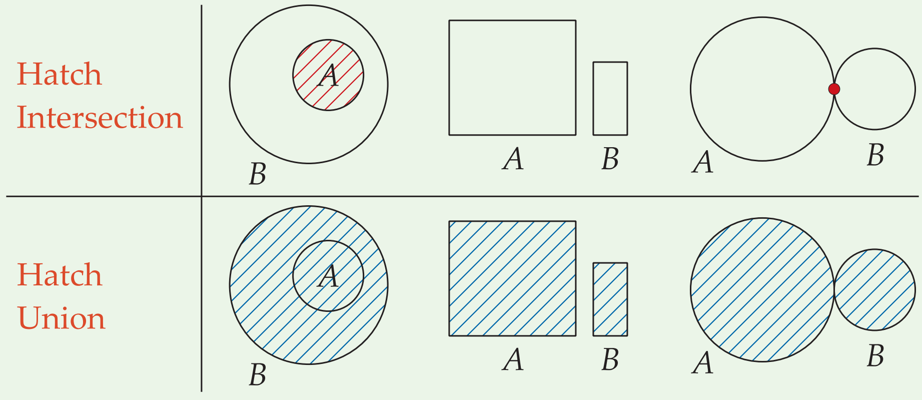


Figure 6.2 Intersection and union of sets

Solution to Activity 2.5

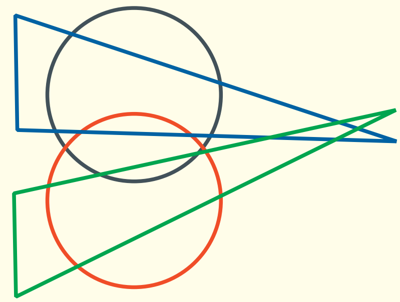


Figure 6.3 Mutually intersecting sets

Solution to Activity 2.6

Note that $A - B = \varnothing$ in the left figure.

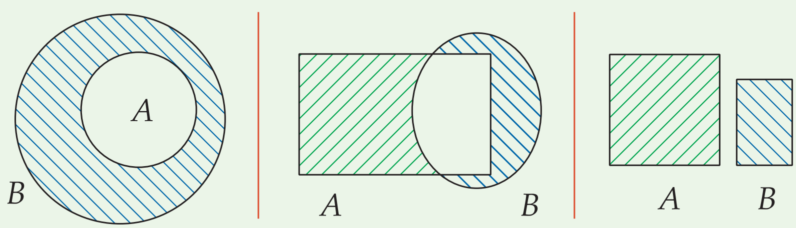


Figure 6.4 Subtraction of two sets

Solution to Activity 2.7

The set $A \cap B$ is hatched in brown, $B \cap C$ in purple, and $C \cap A$ in yellow. The set $A \cap B \cap C$ is hatched with a mixture of brown, purple, and yellow in the center of Figure 6.5.

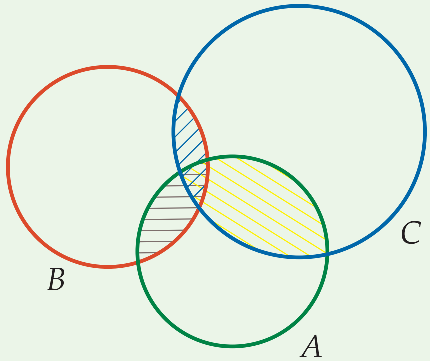


Figure 6.5 Union of three sets

Solution to Activity 2.8

The set $A \cup B$ is hatched in purple, $B \cup C$ is hatched in liver, and $C \cup A$ is hatched in Greenish blue. The set $A \cup B \cup C$ consists of the area in Figure 6.5 shaded with at least one of the above colors.

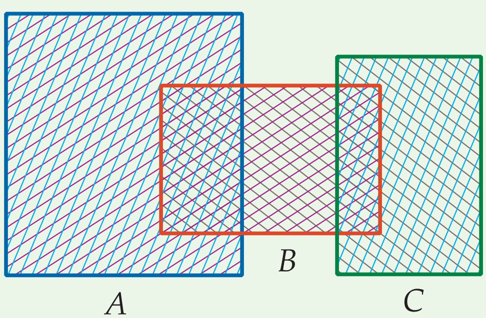


Figure 6.6 Union of three sets

Solution to Activity 2.9

1.

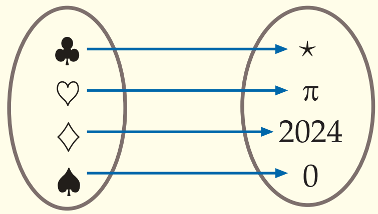


Figure 6.7 A one-to-one correspondence

2.

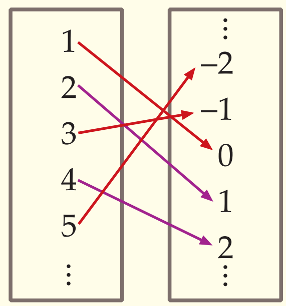


Figure 6.8 A one-to-one correspondence

3. It is not possible to establish any one-to-one correspondence between a finite set and an infinite set. One can refer to the story of **Hotel Infinity** to understand what might happen when we try to accommodate a new guest entering a hotel where all rooms are already occupied; see `https://www.youtube.com/watch?v=Uj3_KqkI9Zo`.

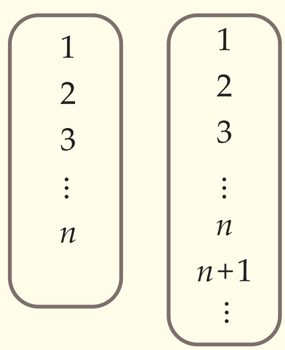


Figure 6.9 There is no one-to-one correspondence

Solution to Activity 2.10

Items 2, 3, 5, 6, 7, and 8 are finite sets, while the rest of the items are infinite sets.

Solution to Activity 2.11

Both sets are infinite since there is a distinct number c between every two numbers a and b (for example, let $c = (a+b)/2$).

6.3 Hints and solutions for Chapter 3

Solution to Activity 3.1

In Figure 6.10,

1. black curve,
2. blue curve,
3. green curve.

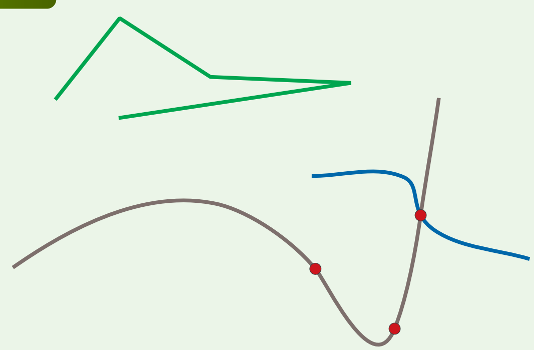


Figure 6.10 Curves through points

Hint for Activity 3.2

You may refer to https://krazydad.com/mazes/.

Solution to Activity 3.3

1. No, it is a simple closed curve.
2. The pink closed curve intersects the green open curve in Figure 6.11.

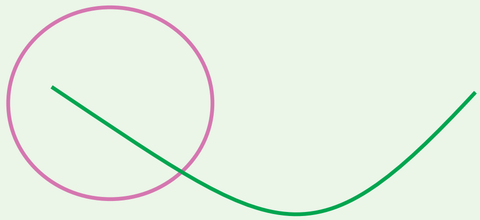


Figure 6.11 Intersecting closed and open curves

3. The orange and blue closed curves intersect each other in Figure 6.12.

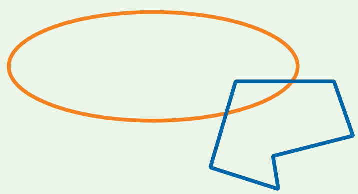


Figure 6.12 Intersecting closed curves

Solution to Activity 3.4

In Figure 6.13,

1. green curve,
2. red curve.

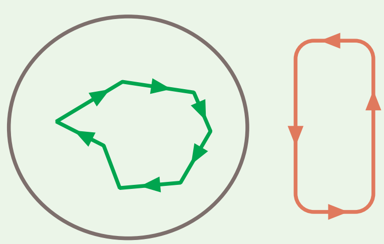


Figure 6.13. oriented simple closed curves

Solution to Activity 3.5

The required curves are shown in Figure 6.14. More precisely,

1. the yellow curve,
2. the green curve,
3. the black curve,
4. the pink curve,
5. the blue curve,
6. the gray curve,
7. the green curve,
8. the red curve.

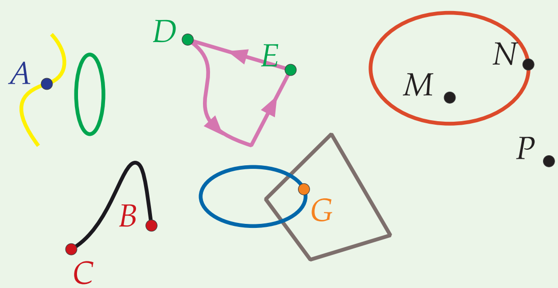


Figure 6.14 Points and curves

Solution to Activity 3.6

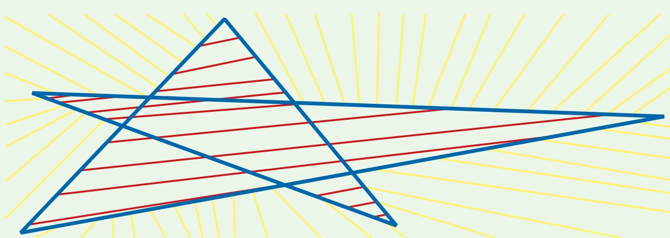


Figure 6.15 Interior and exterior of a curve

Solution to Activity 3.7

The black curve in Figure 6.16.

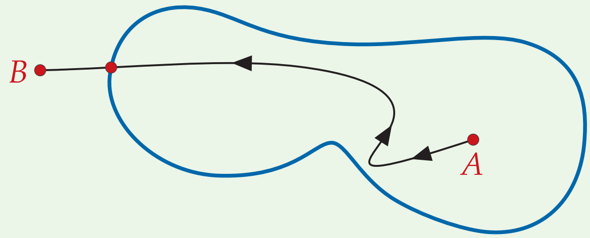


Figure 6.16 Illustrating the Jordan curve theorem

Solution to Activity 3.8

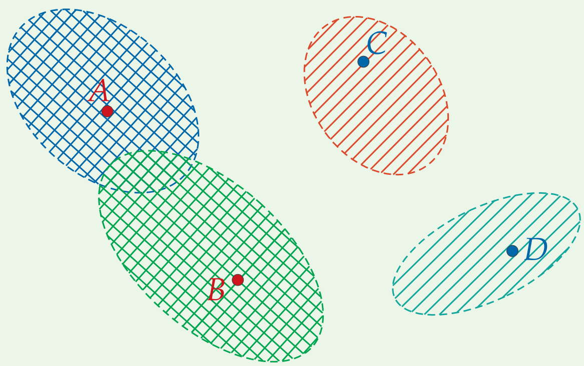


Figure 6.17 Neighborhoods of given points

Solution to Activity 3.9

Interior points, exterior points, and boundary points are given in red, blue, and yelllow, respectively in Figure 6.18.

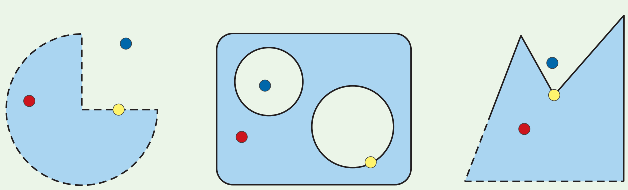


Figure 6.18 Interior points, exterior points, and boundary points

Solution to Activity 3.10

Figure 6.19 Interior, exterior, and boundaries of specific sets

Solution to Activity 3.11

It is important to ensure that each neighborhood is entirely contained in the corresponding blue region.

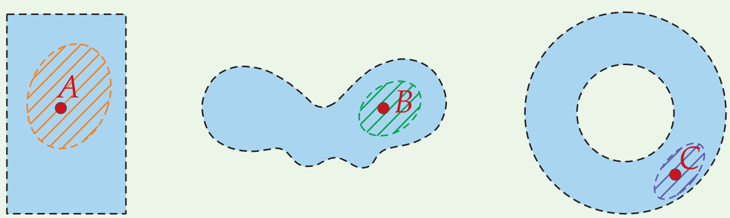


Figure 6.20 Open regions

Solution to Activity 3.12

For any red region in Figure 6.21, there is no neighborhood around the given point that is completely contained in the region.

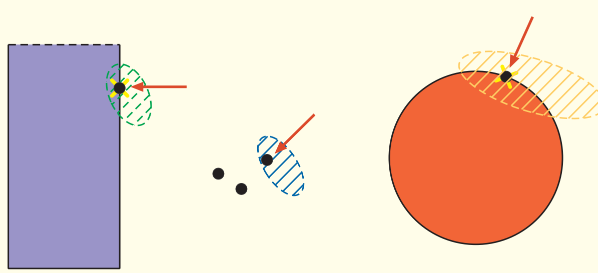


Figure 6.21 Nonopen regions

Solution to Activity 3.13

Consider a point P in the interior of the orange-colored region. There is a yellow-hatched neighborhood of P that is entirely contained in the orange-colored region. Each point A in this yellow-hatched neighborhood is also an interior point of the neighborhood, and therefore of the orange-colored region itself. Consequently, the yellow-hatched neighborhood is contained in the interior of the orange-colored region. Thus, the interior of this region is open, as shown in Figure 6.22. One can similarly prove that the exterior is also open.

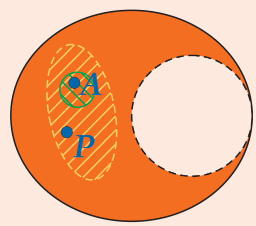


Figure 6.22 Interior and the exterior of a region

Solution to Activity 3.14

The boundaries of regions are depicted in a darker color in Figure 6.23. These regions contain their boundaries, making them closed.

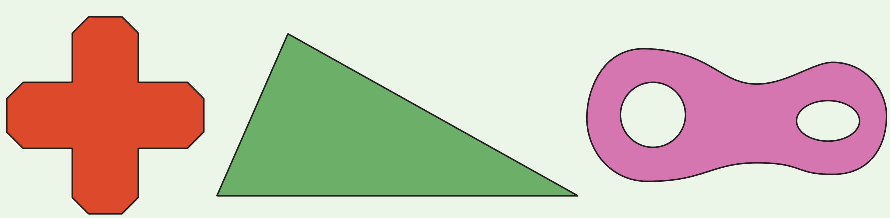


Figure 6.23 Closed regions

Solution to Activity 3.15

In each region, we observe that every neighborhood of the given point intersects both the region and its complement (blue points). Hence these points are considered boundary points but they do not belong to the region.

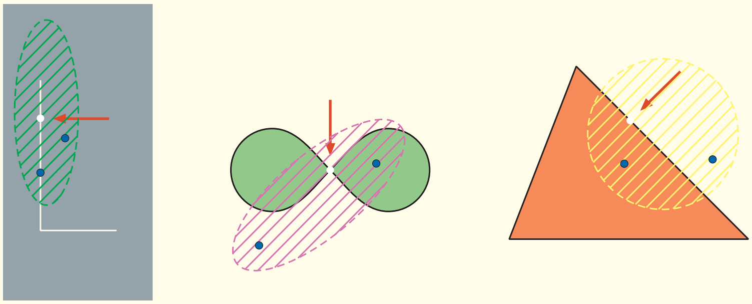


Figure 6.24 Nonclosed regions

Solution to Activity 3.16

Region 1 is open, Regions 2, 3, 5, and 6 are closed, and Region 4 is neither open nor closed.

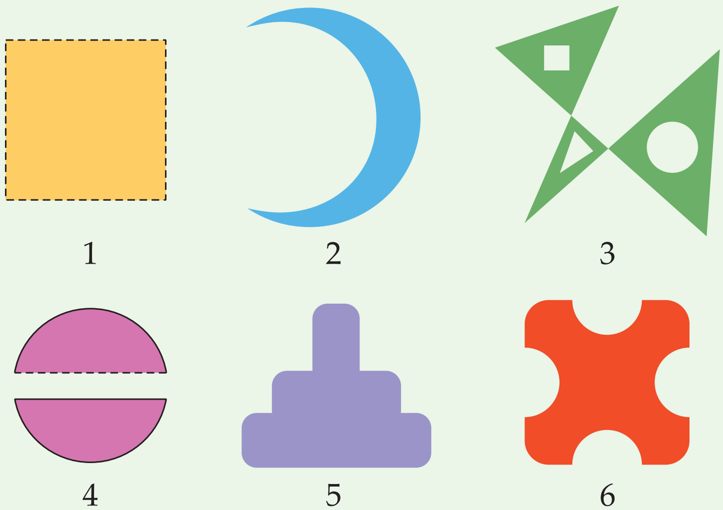


Figure 6.25 Types of regions

Solution to Activity 3.17

1. Suppose G_n represents the green band shown in Figure 6.26 with a width of $\frac{2}{n}$. Hence, G_1 has a width of $\frac{2}{1}$, G_2 has a width of $\frac{2}{2}$, G_3 has a width of $\frac{2}{3}$, and so on. The intersection of $G_1, G_2, \ldots$ is a line (the Y-axis), which is not open in the plane.

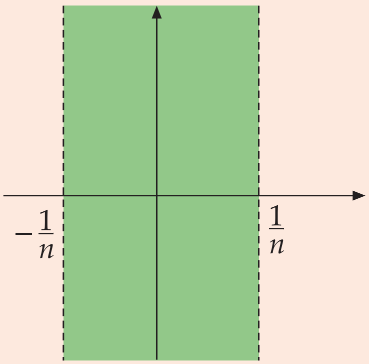


Figure 6.26. The intersection of an arbitrary number of open sets may be not open.

2. Suppose F_n denotes the pink band shown in Figure 6.27 starting from the point $-1+\frac{1}{n}$ and ending at the point $1-\frac{1}{n}$. The union of $F_1, F_2, \ldots$ forms the open band from -1 to 1, which is not closed in the plane.

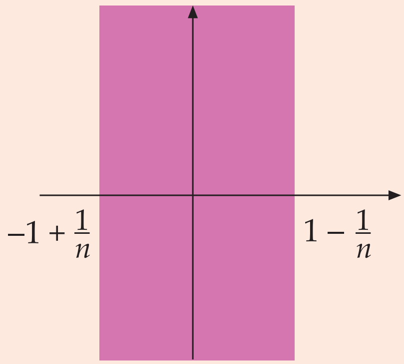


Figure 6.27. The union of an arbitrary number of closed sets may be not closed.

Solution to Activity 3.18

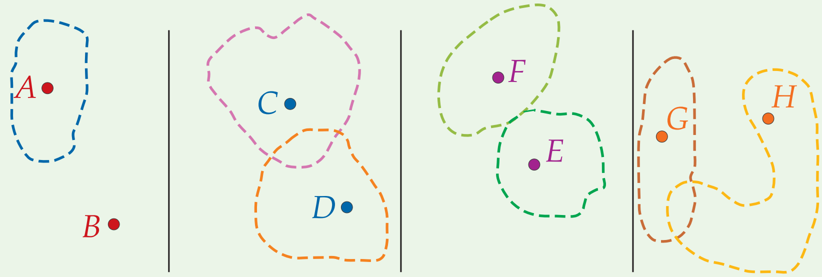


Figure 6.28 Separating points by neighborhoods

Solution to Activity 3.19

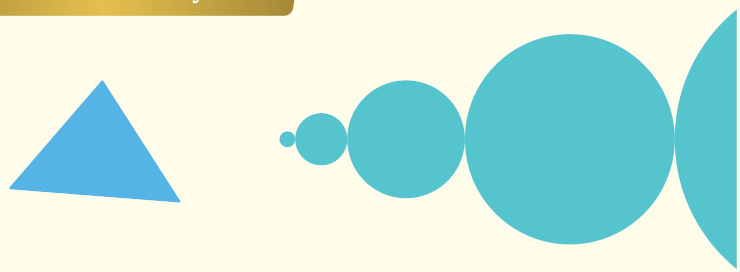


Figure 6.29 Bounded and unbounded regions in the plane

Solution to Activity 3.20

Figure 6.30 A connected region (left) and a disconnected region (right)

Solution to Activity 3.21

Regions 1 and 2 are connected and regions 3 and 4 are disconnected.

Solution to Activity 3.22

In Figure 6.31, the left region has one connected component, while the right region has two connected components.

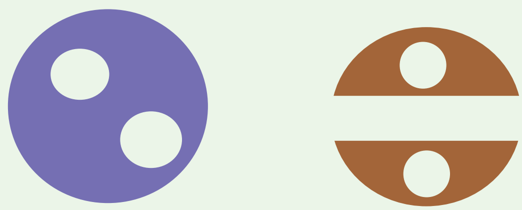


Figure 6.31 Examples of connected and disconnected regions

Solution to Activity 3.23

A curve can be considered as a region in the plane. In Figure 6.32, the region on the left is compact while the region on the right is not compact because it is not closed (the identified point is a boundary point that does not belong to the region).

Figure 6.32 Examples of compact and noncompact sets

Solution to Activity 3.24

The blue and green regions are connected since they are both closed and bounded. The middle region, colored in red, is not compact because there are boundary points at the bottom side that do not belong to the shape.

Solution to Activity 3.25

1. A basketball,
2. a mug,
3. the space,
4. a broken key.

6.4 Hints and solutions for Chapter 4

Hint for Activity 4.1

Take a regular rubber band. Then twist and release it.

Hint for Activity 4.2

Triangles, rectangles, and circles are topologically equivalent because they can be continuously transformed into each other without altering their topological properties.

Hint for Activity 4.3

This activity demonstrates the concept that shapes can undergo continuous transformations while remaining equivalent in topology.

Solution to Activity 4.4

The shape "i " has a component number of 2 (since it is composed of two separate pieces), while the shape "u" has a component number of 1 (since it is a connected shape). Therefore, these shapes are not (topologically) equivalent.

Solution to Activity 4.5

The red shape has a component number of 3, while the blue shape has a component number of 2. Thus, these shapes are not (topologically) equivalent.

Solution to Activity 4.6

The red shape has a disconnecting point of index 3, as shown in Figure 6.33, while the green shape does not have such a point.

Figure 6.33 Application of disconnecting points

Solution to Activity 4.7

The letter X has a disconnecting point of index 4 (red point), while Y and Z do not have such points. The letter Y has a disconnecting point of index 3 (red point) but X and Z do not have such points.

Solution to Activity 4.8

The letter "A" has two disconnecting points of index 3, as shown in Figure 4.15, but the letter "E" only has one such point. Thus, they are not (topologically) equivalent.

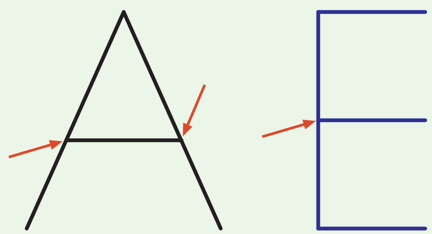


Figure 6.34 Curves having different index of disconnecting point

Solution to Activity 4.9

The shape on the left has a genus number of 0, while the shape on the right has a genus number of 2. Thus, these shapes are not topologically equivalent.

Solution to Activity 4.10

The genus numbers of brick, fork, glass, and ring are 10, 0, 0, and 1, respectively.

Solution to Activity 4.11

The winding numbers of the closed curve with respect to the points A, B, C, D, and E are -1, +2, -1, +1, and +1, respectively.

Hint for Activity 4.12

Consider a curve that spirals around a point. The curve approaches the point arbitrarily but never actually touches it. In this case, the curve effectively wraps around the point an infinite number of times as it approaches.

Solution to Activity 4.13

$V - E + F = 5 - 8 + 5 = 2$

Solution to Activity 4.14

The graph on the right has a point, shown in Figure 6.35, whose degree is 6, while the graph on the left has no such a point. Therefore, these graphs are not topologically equivalent.

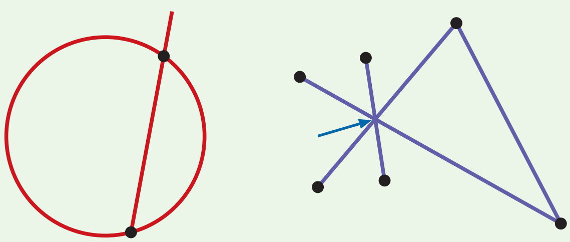


Figure 6.35 Degree of a vertice

Hint for Activity 4.15

Some good sources for this activity are https://www.youtube.com/watch?v=-kA1_h1dZ58 and https://www.youtube.com/watch?v=X1QOipIVFPk.

Hint for Activity 4.16

Topologically speaking, having two holes in a soccer ball implies that there are two distinct regions where the surface of the ball is cut through or punctured. If we imagine these holes as being close enough to each other, we can consider them as a single, elongated hole. A stretching process turns the soccer ball into a cylindrical shape.

Solution to Activity 4.17

A cube has 8 vertices, 12 edges, and 6 faces. Hence $V - E + F = 8 - 12 + 6 = 2$ for a cube. For a pyramid, we have $V - E + F = 5 - 8 + 5 = 2$.

Hint for Activity 4.18

Add more edges by connecting the inner and outer squares of any two adjacent faces.

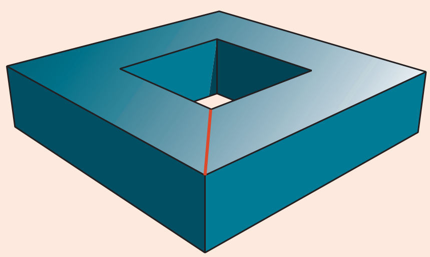


Figure 6.36 Cube with a hole

Hint for Activity 4.19

Refer to https://en.wikipedia.org/wiki/Art_gallery_problem

Hint for Activity 4.20

In three-dimensional space, it is impossible to attach a circle to the m-boundary of a Möbius strip without the surface intersecting itself.

Hint for Activity 4.21

Visit https://www.youtube.com/watch?v=AAsICMPwGPY.

6.5 Hints and solutions for Chapter 5

Solution to Activity 5.1

(a) The projection of a sphere (or ball) onto the two-dimensional plane is the set of points on and inside a circle.

(b) The projection of a hypersphere (or hyperball) onto the three-dimensional space is a sphere (or ball).

Hint for Activity 5.2

To create the Chefalo, take a rope, and form a simple loop. Next, twist the loop so that it forms a helix-like structure, and finally tie off the ends.

Hint for Activity 5.3

The symmetry properties of a knot can be used to explain topological differences. Some knots exhibit rotational symmetry or reflectional symmetry. These symmetry properties are topological invariants that can distinguish between different knot types.

Solution to Activity 5.4

1.

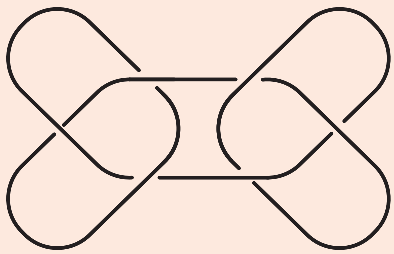


Figure 6.37. Knot diagram of granny knot

2.

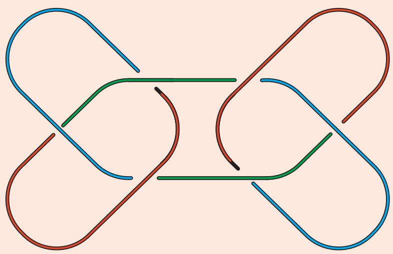


Figure 6.38. Tricolorability of granny knot

Solution to Activity 5.5

1.

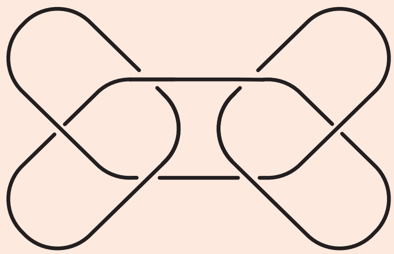


Figure 6.39. Knot diagram of square knot

2.

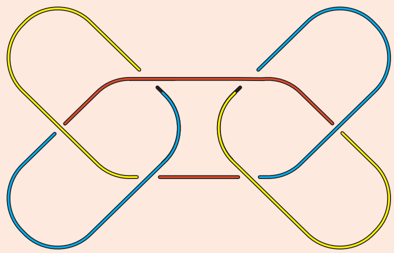


Figure 6.40. Tricolorability of square knot

Hint for Activity 5.6

Show that any two points in this set are separated by at least one removed interval.

Solution to Activity 5.7

Yes, it satisfies all conditions of topology. This topology is called the Sierpiński topology.

Solution to Activity 5.8

No, $\{1\} \cup \{2\}$ is not in $\{\emptyset, \{1\}, \{2\}\}$.

Solution to Activity 5.9

Yes, it satisfies all conditions of topology.

Hint for Activity 5.10

Some of the required topologies are discrete and indiscrete topologies. When dealing with small values of n, one can manually count the number of topologies. However, for larger values of n, it becomes computationally intensive.

Note

AI-based tools have been used during proofreading.

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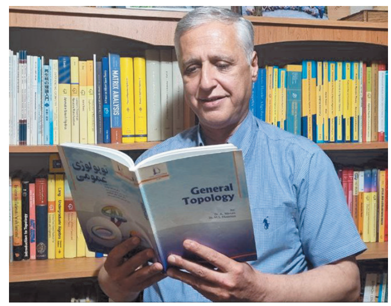
About the Author



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